MISN-0-12

SPECIAL RELATIVITY: THE LORENTZ TRANSFORMATION, THE VELOCITY ADDITION LAW

$$\frac{1}{2} + \frac{1}{2} = \frac{4}{5}$$

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

SPECIAL RELATIVITY: THE LORENTZ TRANSFORMATION,

THE VELOCITY ADDITION LAW

by

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Title: **Special Relativity: The Lorentz Transformation, The Velocity Addition Law**

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Input Skills:

1. Vocabulary: invariance, observer, relative linear motion, velocity addition law (MISN-0-11).
2. Given an object's motion relative to one observer, and that observer's motion relative to a second observer, describe the object's motion relative to the second observer (MISN-0-11).

Output Skills (Knowledge):

K1. Vocabulary: Galilean transformation, Lorentz transformation, relativistic velocity addition law.

K2. Derive the Galilean transformation from the non-relativistic velocity addition law and show that it is invariant under interchange of the observer labels. Explain why one intuitively believes that this should be so.

K3. Explain clearly how the observed constancy of the speed of light with respect to all observers shows that the Galilean transformation must be erroneous.

K4. Given the Lorentz transformation, derive the relativistic velocity addition law from it.

K5. Show that the relativistic velocity addition law is in agreement with the fact that the speed of light is the same for all observers.

K6. Show that, for low enough relative speeds, the difference between the relativistic and non-relativistic velocity addition laws becomes unobservable.

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1. The Galilean Transformation

1a. Applicable Situations. The Galilean transformation applies to the physics of "everyday" life, where objects move at "normal" speeds. It follows from the "ordinary" velocity addition law.

1b. Velocity Addition Law. Consider an object O with (one-dimensional) velocity $v_{\rm OA}(t)$ as determined by ("relative to") observer A and velocity $v_{\rm OB}(t)$, relative to observer B. The ordinary velocity addition law states that:

$$v_{\rm OB}(t) = v_{\rm OA}(t) - v_{\rm BA}\,, \tag{1}$$

where $v_{\rm BA}$ is the velocity of observer B relative to observer A.[1] Velocity $v_{\rm BA}$ is restricted to being independent of time for the purposes of this unit.

1c. Transformation Derivation. In order to obtain the Galilean transformation, we must get x's into the equation. This is easy since

$$v_{\rm OB}(t_{\rm B}) \equiv dx_{\rm OB}(t_{\rm B})/dt_B\,,$$

and

$$v_{\rm OA}(t_{\rm A}) \equiv dx_{\rm OA}(t_{\rm A})/dt_{\rm A}\,, \tag{2}$$

where t_A is the time as measured on A's clock, while $t_{\rm B}$ is the time as measured on B's clock. Then $v_{\rm OB}(t_{\rm B})$ is the velocity of O as seen by B at the time $t_{\rm B}$.

[1] In one-dimensional problems all vectors can be arranged to point along a single axis. Then in vector equations the same unit vector is common to all terms so it can be factored out and dropped. That leaves the vector components along that single axis, for which the axis subscript can also be dropped. These quantities, such as $v_{\rm OB}$, $v_{\rm OA}$, and $v_{\rm BA}$ in Eq. (1) are not vector *magnitudes*, which would always be positive; instead they are vector *components* which can have negative as well as positive numerical values.

The double subscript notation is developed in more detail in "Relative Linear Motion, Frames of Reference" (MISN-0-11).

In words, Eq. (2) shows that the velocity of O as observed by A is the time rate of change of the position of the object, as observed by A with A's measuring devices (clocks and rulers). Substituting Eq. (2) into Eq. (1), and integrating with respect to $t_{\rm A}$:

$$\int_{t_0}^{t} \frac{dx_{\rm OB}(t_{\rm B})}{dt_{\rm B}}dt_{\rm A} = \int_{t_0}^{t} \frac{dx_{\rm OA}(t_{\rm A})}{dt_{\rm A}}dt_{\rm A} - \int_{t_0}^{t} v_{\rm BA}dt_{\rm A}\,,$$

where the left-hand-side (LHS) is not integrable until we put in the Galilean time transformation, $t_{\rm B} = t_{\rm A}$. This is meant to be obvious, in the sense that "everyone knows" that A's clocks can all be synchronized with B's. For convenience, we replace $t_{\rm B}$ and $t_{\rm A}$ by the single symbol t. Integrating, we get

$$x_{\rm OB}(t) - x_{\rm OB}(t_0) = x_{\rm OA}(t) - x_{\rm OA}(t_0) - v_{\rm BA}(t - t_0)\,.$$

Again in the interest of simplicity we set $t_0 = 0$ and set up A's and B's x-axis origins at such places that they coincide at time zero. Then the two observers' measured positions of the object will agree at that time and so:

$$x_{\rm OB}(t) = x_{\rm OA}(t) - v_{\rm BA}t\,. \tag{3}$$

This equation, along with $t_{\rm B} = t_{\rm A} \equiv t$ is the Galilean transformation.

1d. Invariance Under Label Interchange. To show that Eq. (3) is invariant under interchange of observer labels, simply replace A by B and B by A everywhere in the equation and show that by a little manipulation you can get Eq. (3) back again. One intuitively feels that the laws of nature should be independent of which label one gives to which observer.

1e. Predicted Light Speed Varies. The velocity addition law associated with the Galilean transformation is, of course, the same one we started with: $v_{\rm OB} = v_{\rm OA} - v_{\rm BA}$. We now apply this addition law to the case where the object O is the front end of a beam of light (L). Hence $v_{\rm OB} = v_{\rm LB}$ is the velocity of light measured by observer B, $v_{\rm OA} = v_{\rm LA}$ is the velocity measured by observer A and the two velocities are related by:

$$v_{\rm LB} = v_{\rm LA} - v_{\rm BA}\,.$$

Notice that, if A and B are moving relative to each other ($v_{\rm BA} \neq 0$), then each sees a different speed of light. This is, of course, the "common sense" result.

2. The Lorentz Transformation

2a. Observed Constancy of Light Speed. Toward the end of the nineteenth century, this aspect of the Galilean transformation—the non-constancy of the speed of light—came into conflict with Maxwell's formulation of the theory of electricity and magnetism and with the experimental tests of that formulation. The theoretical and experimental results pointed toward one conclusion - that the velocity of light must be the same for all observers, regardless of their motion relative to each other, that is, $v_{\rm LB} = v_{\rm LA} = c$, a constant.

The Galilean transformation cannot give this result; the one that does is the Lorentz transformation.

2b. The Transformation. The Lorentz transformation is given by:

$$x_{\rm OB} = k\left(x_{\rm OA} - v_{\rm BA} t_{\rm A}\right)\,, \tag{4}$$

$$t_{\rm B} = k\left(t_{\rm A} - v_{\rm BA} x_{\rm OA}/c^2\right)\,, \tag{5}$$

$$k \equiv \left(1 - v_{\rm BA}^2/c^2\right)^{-1/2}\,. \tag{6}$$

The Lorentz transformation has the peculiar features that the length of an object such as a meter stick is contracted and the rate of a clock is slowed when observed from a moving reference frame.[2]

Note, that, in the limit that $v_{\rm BA} << c$, the Lorentz transformation reduces to the Galilean transformation.

3. The Velocity Addition Law

3a. Derivation. The relativistic velocity addition law can be easily derived from the Lorentz transformation. First,

$$v_{\rm OB} = \frac{dx_{\rm OB}}{dt_B} = \frac{dx_{\rm OB}}{dt_A}\cdot\frac{dt_A}{dt_B} = \frac{dx_{\rm OB}}{dt_A}\cdot\left(\frac{dt_B}{dt_A}\right)^{-1}\,. \tag{7}$$

Differentiating Eq. (4) with respect to $\rm t_{OA}$ gives:

$$\frac{dx_{\rm OB}}{dt_A} = k\left(\frac{dx_{\rm OA}}{dt_A} - v_{\rm BA}\frac{dt_A}{dt_A}\right) = k(v_{\rm OA} - v_{\rm BA})\,.$$

[2]See "The Length Contraction and Time Dilation Effects of Special Relativity" (MISN-0-13). This unit culminates in an examination of the "twin paradox": The Lorentz transformation predicts that, if twin A is moving, A will age more slowly than will stationary twin B. However, twin A could equally well say that it is twin B who is moving, albeit in the opposite direction ($v_{\rm BA} = -v_{AB}$), and therefore it is B who should age more slowly.

And differentiating Eq. (5) with respect to t_A gives:

$$\frac{dt_B}{dt_A} = k\left(\frac{dt_A}{dt_A} - v_{\rm BA}\frac{dx_{\rm OA}}{dt_A}/c^2\right) = k(1 - v_{\rm BA}v_{\rm OA}/c^2)\,.$$

So substituting these into Eq. (7) gives:

$$v_{\rm OB} = k(v_{\rm OA} - v_{\rm BA})\left[k(1 - vBAvOA/c^2)\right]^{-1}\,,$$

so:

$$v_{\rm OB} = \frac{v_{\rm OA} - v_{\rm BA}}{1 - v_{\rm BA}v_{\rm OA}/c^2}\,, \tag{8}$$

which is the relativistic velocity addition law.

3b. Velocity Much Less Than the Speed of Light. Note that here, too, in the limit that $v_{\rm BA} << c$, our result reduces to the "ordinary" velocity addition law.

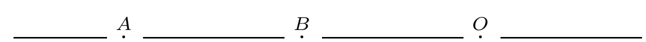
3c. Predicting Constant Light Speed. We can verify that the speed of light is the same for all observers. If we put $v_{\rm OA} = v_{\rm LA} = c$ into Eq. (8) then

$$v_{\rm LB} = \frac{c - v_{\rm BA}}{1 - v_{\rm BA}c/c^2} = \frac{c - v_{\rm BA}}{1 - v_{\rm BA}/c} = c\,.$$

So, both A and B see light as having the same speed.

3d. Reduction to Galilean Law. We have pointed out that, in the limit that $v_{\rm BA}$ is much less than the speed of light, the relativistic velocity addition law reduces to the ordinary velocity addition law. This can be demonstrated by putting "everyday" speeds into the velocity addition laws and comparing the results.

3e. A Numerical Example.



Consider, for instance, the following situation. Observer A sees object O moving in the $-x$-direction, toward him/her, at a speed of 1000 m/s (2237 mi/hr), and A also sees observer B moving in the $+x$-direction, away from her/him, at 1000 m/s. According to the ordinary velocity addition law, observer B will therefore see object O as having velocity:

$$v_{\rm OB} = v_{\rm OA} - v_{\rm BA} = (-1000\,{\rm m/s}) - 1000\,{\rm m/s} = -2000\,{\rm m/s}\,.$$

That is, O is seen by B as approaching him/her at 2000 m/s. Using the relativistic velocity addition law, B sees O as having velocity

$$\begin{aligned} v_{\rm OB} &= \frac{v_{\rm OA} - v_{\rm BA}}{1 - v_{\rm BA} v_{\rm OA}/c^2} \\ &= \frac{(-1000\,{\rm m/s}) - 1000\,{\rm m/s}}{1 - (1000\,{\rm m/s})(-1000\,{\rm m/s})/(3 \times 10^8\,{\rm m/s})^2} \\ &= \frac{-2000\,{\rm m/s}}{1 + \dfrac{10^6\,{\rm m^2/s^2}}{9 \times 10^{16}\,{\rm m^2/s^2}}} \simeq \frac{-2000\,{\rm m/s}}{1 + 1.1 \times 10^{-11}} \\ &= \frac{-2000\,{\rm m/s}}{1.000000000011} = -1999.9999978\,{\rm m/s}\,. \end{aligned}$$

The difference between the two results is of the order of one part in a billion. So, for this case, the difference between the results of the ordinary velocity addition law and the relativistic velocity addition law are too small to be measurable in the everyday world.

▷ A spaceship is approaching the earth at a speed of 0.9000 c. A cyclotron mounted on the spaceship sends out a beam of protons with speed 0.9000 c relative to the spaceship. Show that an observer on the earth sees the beam of protons approaching her or him with a speed of 0.9945 c.

Acknowledgments

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Glossary

- **Galilean transformation**: the transformation between frames of reference that applies when the relative linear velocity of one reference frame with respect to the other is much less than the speed of light.

- **Lorentz transformation**: the transformation between frames of reference that applies when the relative linear velocity of one reference frame with respect to the other has any value whatsoever.

- **relativistic velocity addition law**: a statement of the way in which velocities transform from one constant-speed frame of reference to another.

MODEL EXAM

1. See Output Skills K1-K6 in this module's *ID Sheet*. The actual exam may contain any number of these skills.

Brief Answers:

1. See this module's *text*.