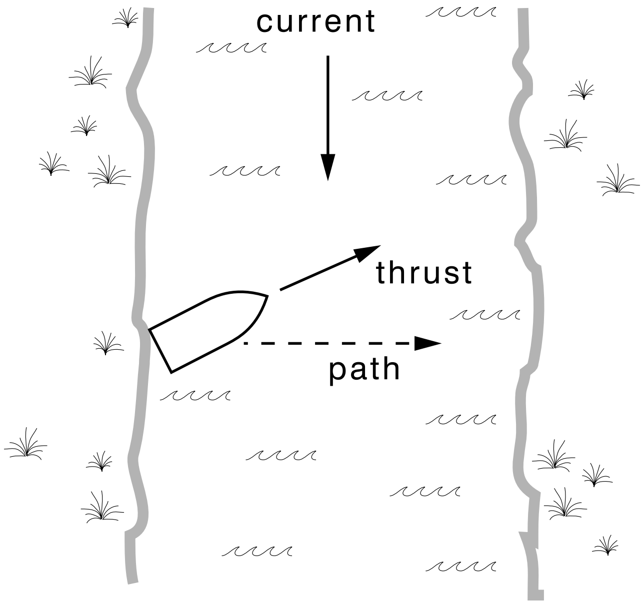
MISN-0-11

RELATIVE LINEAR MOTION, FRAMES OF REFERENCE



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

RELATIVE LINEAR MOTION, FRAMES OF REFERENCE

by

Peter Signell and William Lane

Title: **Relative Linear Motion, Frames of Reference**

Author: Peter Signell and William C. Lane, Department of Physics

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Input Skills:

1. Add and subtract vectors using their Cartesian components (MISN-0-2).
2. Given a vector described either in terms of Cartesian components or magnitude and direction, determine the other vector description (MISN-0-2).
3. Given an object's position as a function of time, find its velocity and acceleration as functions of time (MISN-0-8).
4. Detect errors in symbolic equations by checking dimensions (MISN-0-8).

Output Skills (Knowledge):

K1. Vocabulary: frame of reference, observer.

Output Skills (Problem Solving):

S1. Given the position vectors of both an object and an observer as seen by a second observer, find the position, velocity and acceleration vectors of the object as seen by the first observer. Use suitable notation for labelling observers and observed. Sketch any of the vectors with respect to any frame of reference, as requested.

S2. Given a kinematical equation containing both symbols and numbers plus a word description of the units involved, properly insert the units into the equation and carry out any appropriate units algebra.

S3. In any given kinematical problem, determine whether $a = dv/dt$ is valid.

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Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

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1. Applications of Relative Motion

1a. Navigation. An application of relative motion which immediately comes to mind is that of navigation through currents of water and air. Consider two observers and an object, where each observer sees both the other observer and the object as moving. Given the equation of motion of the object as seen by one observer, what equation of motion will the other observer see for it? Here the equation of motion set by the pilot, with respect to the fluid through which he moves, must be such as to produce the correct equation of motion with respect to the land. Can you tell which are the two "observers" and which is the "object?" Can you transform the desired land-based equation into the fluid-based one needed for navigation?

1b. Solar System Dynamics. For another application, consider the apparent motions of the planets across the night sky. Since very ancient times, the positions of the planets at various times had been carefully recorded. In about 140 A.D., from this vast set of numbers, the planetary equations of motion were finally deduced by Ptolemy. These equations were so excruciatingly complicated, however, that the forces which would produce them must have seemed incomprehensible. Then in 1543, in what must have been one of the most thrilling discoveries in history, Copernicus transformed the equations of motion to the way they would be observed from the sun and thereupon found them to be trivially simple. This eventually spurred Newton to discover the elegantly simple universal laws of motion and of gravitation, and to invent calculus. From that day to this, a fundamental assumption of physicists has been that all of the forces of nature will be found to be elegantly simple when properly expressed.

2. Double Subscript Notation

2a. Relative Position Vectors. Double subscript notation is a simple and consistent method of dealing with the relative positions of two objects.

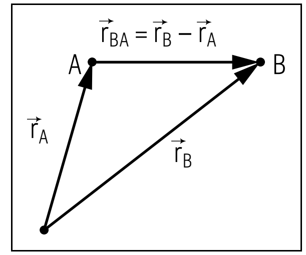

Figure 1. Vector interpretation of double subscript notation.

If $\vec r_A$ is the position vector of object A and $\vec r_B$ is the position vector of object B in a given coordinate system, then the position vector of B relative to A is defined by

$$\vec r_{BA} = \vec r_B - \vec r_A\,. \tag{1}$$

Notice that $\vec r_{BA}$ is the vector which, when added to $\vec r_A$, gives $\vec r_B$:

$$\vec r_A + \vec r_{BA} = \vec r_B\,. \tag{2}$$

That is, if you start at the position of object A, and move along $\vec r_{BA}$, you end up at the position of object B (see Fig. 1).

2b. The Order of the Subscripts. The order of the subscripts of a relative position vector determines the direction of the vector. The vector $\vec r_{BA}$ is directed from object A to object B. The vector $\vec r_{AB}$, from B to A, would obviously point in the opposite direction, and have exactly the same length as $\vec r_{BA}$, so

$$\vec r_{AB} = -\vec r_{BA}\,. \tag{3}$$

2c. Velocity and Acceleration from Relative Position. Once the relative position has been determined, it is simple to get the velocity and acceleration of B relative to A, just by differentiating Eq. (1):

$$\vec v_{BA} = \frac{d}{dt}(\vec r_{BA}) = \frac{d}{dt}(\vec r_B - \vec r_A) = \frac{d}{dt}\vec r_B - \frac{d}{dt}\vec r_A = \vec v_B - \vec v_A$$

$$\vec a_{BA} = \frac{d}{dt}\vec v_{BA} = \frac{d}{dt}(\vec v_B - \vec v_A) = \frac{d}{dt}\vec v_B - \frac{d}{dt}\vec v_A = \vec a_B - \vec a_A\,,$$

so

$$\vec v_{BA} = \vec v_B - \vec v_A\,, \tag{4}$$

$$\vec a_{BA} = \vec a_B - \vec a_A\,. \tag{5}$$

Notice that the mathematical relationships between $\vec v_{BA}$, $\vec v_A$, and $\vec v_B$, and between $\vec a_{BA}$, $\vec a_A$, and $\vec a_B$ are identical to those between $\vec r_{BA}$, $\vec r_A$, $\vec r_B$. For example, if you start out having velocity $\vec{v}_A$ (or acceleration $\vec{a}_A$), give yourself the additional velocity $\vec{v}_{BA}$ (or acceleration $\vec{a}_{BA}$), you end up having velocity $\vec{v}_B$ (or acceleration $\vec{a}_B$).

2d. Subscript Reversal on $\vec{v}$ and $\vec{a}$. It is also easy to see that the velocity of A relative to B, $\vec{v}_{AB}$, and the acceleration of A relative to B, $\vec{a}_{AB}$, are related to $\vec{v}_{BA}$ and $\vec{a}_{BA}$ in just the same way as $\vec{r}_A B$ is related to $\vec{r}_{BA}$:

$$\vec{v}_{AB} = -\vec{v}_{BA}\,, \tag{6}$$

$$\vec{a}_{AB} = -\vec{a}_{BA}\,. \tag{7}$$

2e. An Example. As an example of the velocity addition law, Eq. (4), suppose you are driving down a highway at 45 mph when a state trooper approaches you from the rear at 65 mph. If the trooper has a radar gun trained on you, what will it read? Using Eq. (4), we find the answer to be 20 mph. *Help: [S-3]* Perhaps the answer was obvious to you without using Eq. (4).

Now suppose the trooper is traveling along a road that makes an angle with the road on which you are traveling. What will the radar gun read then? Perhaps it is obvious that Eq. (4) is needed in order to determine the result in this case. *Help: [S-4]*

3. Frames of Reference

3a. "Observer" Labels. Sometimes it may be easier to deal solely with relative quantities like $\vec{r}_{AB}$ instead of "absolute" ones like $\vec{r}_A$. To

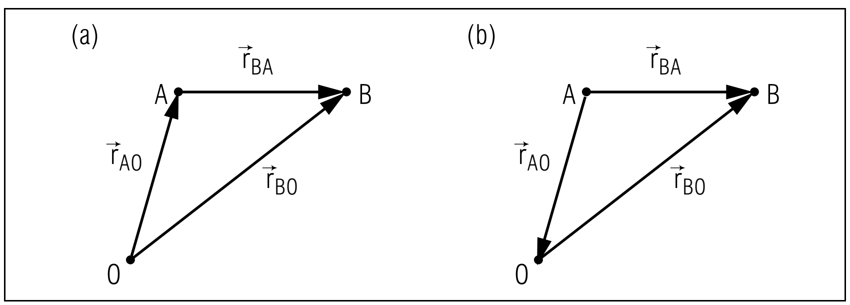

Figure 2. Relative position vectors defined for various observer labels where $\vec{r}_{BA}$ is (a) the difference between two vectors, or (b) the sum of two vectors.

do this, we can simply place an observer O at the origin of the coordinate system and measure all the vector quantities relative to that observer. The coordinate system is called the observer's "frame of reference." Thus $\vec{r}_A$ becomes $\vec{r}_{AO}$, $\vec{v}_A$ becomes $\vec{v}_{AO}$, and so on (see Fig. 2a). Equation (1) then becomes

$$\vec{r}_{BA} = \vec{r}_{BO} - \vec{r}_{AO}\,, \tag{8}$$

or, by applying Eq. (3) to get $\vec{r}_{AO} = -\vec{r}_{OA}$ (see Fig. 2b), we have:

$$\vec{r}_{BA} = \vec{r}_{OA} + \vec{r}_{BO}\,. \tag{9}$$

3b. Checking an Equation's Subscript Sequence. The addition equation, Eq. (9), may be read "A to B equals A to O plus O to B." Note that one reads the subscripts from right to left. This provides a powerful check on the correctness of relative-vector equations, since it is usually quite easy to rearrange terms into the addition form.

3c. Converting From Other Notations. Many scientists use a "prime-unprimed" notation when describing vectors relative to two frames of reference. Vectors in that notation may be converted to double subscript notation by placing observers (say, O and O') at the origins of the reference frames and expressing the vector quantities relative to these observers. For example, consider the "prime-unprimed" velocity addition equation

$$\vec{v} = \vec{v}\,' + \vec{u}\,, \tag{10}$$

where $\vec{v}$ is the velocity of an object in the unprimed reference frame, $\vec{v}\,'$ is the velocity of the object in the primed reference frame, and $\vec{u}$ is the velocity of the primed frame relative to unprimed frame. Equation (10) can be rewritten as

$$\vec{v}_{AO} = \vec{v}_{AO'} + \vec{v}_{O'O}\,, \tag{11}$$

by making the identifications

$$\vec{v} = \vec{v}_{AO} = \text{velocity of } A \text{ relative to } O$$
$$\vec{v}\,' = \vec{v}_{AO'} = \text{velocity of } A \text{ relative to } O'$$
$$\vec{u} = \vec{v}_{O'O} = \text{velocity of } O' \text{ relative to } O\,.$$

The double subscript form is easy to check visually.

Acknowledgments

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Research, through Grant #SED 74-20088 to Michigan State University.

Glossary

- **frame of reference**: a spatial coordinate system that is used to determine the position or describe the motion of some object of interest.
- **observer**: a person making measurements, descriptions, or determinations of some event.

PROBLEM SUPPLEMENT

Problems 6 and 7 also occur in this module's *Model Exam.*

1. Two cars on a straight section of road are traveling parallel to each other. Car 1 has a speed of 60.0 mi/hr and is traveling eastward, while Car 2 has a speed of 45 mi/hr. Calculate the relative velocity of Car 2 with respect to Car 1 when:

   a. the two cars are traveling the same direction;

   b. the two cars are traveling in opposite directions.

2. A speed boat capable of traveling 35 m/s in still water attempts to cross a river 1500 m wide with a current moving at a speed of 28 m/s. Determine the direction the boat must travel relative to the current if the resultant path of the boat is to be straight across the river, perpendicular to the banks. How long will the trip take?

3. A particle Q has a position, measured with respect to coordinate system O, of:

   $$\vec{r} = (6t^2 - 4t)\hat{x} - 3t^3\hat{y} + 3\hat{z}\,.$$

   With respect to another coordinate system O', this same particle's position is:

   $$\vec{r} = (6t^2 + 3t)\hat{x} - 3t^3\,\hat{y} + 3\hat{z}\,.$$

   (Note: all distances are in meters, all times are in seconds.)

   a. Rewrite the two position vector equations with double subscripts and proper units.

   b. Find the time at which the two observers' coordinate axes coincide.

   c. Find the velocity of O' with respect to O. *Help: [S-5]*

   d. Find the distance between O and O' at time $t = 3$ s.

   e. Find the distance between Q and O at time $t = 3$ s.

   f. Find the distance between Q and O' at time $t = 3$ s.

   g. Find the angle between $\vec{r}_{QO}$ and $\vec{r}_{QO'}$ at $t = 3$ s.

   h. Find the angle between $\vec{r}_{QO}$ and $\vec{r}_{QO'}$, as seen looking along the z-axis, at $t = 3$ s.

i. Draw a rough sketch showing $\vec{r}_{QO}$, $\vec{r}_{QO'}$, and $\vec{r}_{OO'}$, as they appear if you are looking along the z-axis toward the origin. Label the positions of Q, O, and O'.

4. An air traffic controller is tracking, via radar, two planes cruising at the same altitude. At a particular time Plane A is 12 miles away, bearing 60° (measured clockwise from north) and is traveling due north at 250 mi/hr. Plane B is 10 miles out, bearing 37° and is traveling due east at 550 mi/hr.

   a. Draw a sketch of the position and velocity vectors of each plane with respect to the controller's position.

   b. Express the position vector of each plane, and the position of plane B relative to plane A, as functions of time.

   c. Will the two planes collide if each plane holds its course? If so, how long after the time of the above initial conditions will the collision take place? Would you recommend this calculation be made by hand calculator, slide rule, or computer?

   d. What is the velocity of plane B relative to plane A?

5. In 1821 H. M. S. Clorinda, a British frigate, attempted to capture the Estrella del Sur, a Spanish schooner. At a particular point of the chase the schooner, well out of range, changed course to cross Clorinda's bow and head for San Juan, Puerto Rico on a NW course (N 45° W) at a speed of 8 knots. Clorinda, heading N 11.25° W at 6 knots, saw the schooner 3 points off the starboard bow (i.e. at a bearing of 33.75° measured clockwise from dead ahead) at a range of 0.25 nautical miles. Maximum cannon shot range was 2 cable lengths (0.20 nautical miles).

   a. Sketch the situation at the time of the schooner's course change. Include the velocities of each ship and the position of the Estrella del Sur relative to Clorinda.

   b. Calculate the velocity of the schooner relative to the frigate.

   c. Express the position of the schooner relative to the frigate as a function of time.

   d. Express the distance of the schooner relative to the frigate as a function of time.

   e. What is the distance of closest approach of the two ships? Will the schooner ever be in range?

Note: nautical mile = 1.0×10^1 cable lengths
knot = nautical mile/hr.

6. You are a small-craft airplane pilot. You have pointed your plane due west and maintained a speed of 120 mi/hr as registered on your air-speed indicator (speed with respect to the air) for two hours. However, there is a wind blowing out of the north-east at 4.0×10^1 mi/hr. Calculate the distance you are off your desired due-west course at the end of the two hours.

7. Relative to the origin of a laboratory reference frame L, an object A's position varies with time according to

$$\vec{r}_A = 5t^2\hat{x} + 8\hat{y}\,,$$

while an observer O sees

$$\vec{r}_A = 7t^2\hat{x} - 3t^{-1}\hat{z}\,.$$

   a. Rewrite the above vector equations, putting in double subscripts and proper units. Assume that all distance quantities are in meters, all time quantities in hours.

   b. Find the speed (magnitude of the velocity) of O relative to L at time t.

   c. Find the magnitude of the acceleration of O relative to L at time t.

   d. Draw a rough sketch of the position and velocity vectors of O, as seen by L at $t = 90$ min, as they would appear looking down the z-axis toward the origin.

Brief Answers:

1. a. 15 mi/hr, westward

   b. 105 mi/hr, westward

2. 143° from the direction of the current. The trip will take 71 seconds.

3. a. $\vec{r}_{QO} = \left[(6\,\text{m/s}^2)t^2 - (4\,\text{m/s})t\right]\hat{x} + (-3\,\text{m/s}^3)t^3\hat{y} + (3\,\text{m})\hat{z}$

   $\vec{r}_{QO'} = \left[(6\,\text{m/s}^2)t^2 + (3\,\text{m/s})t\right]x + (-3\,\text{m/s}^3)t^3\hat{y} + (3\,\text{m})\hat{z}$

   b. $t = 0$

c. $\vec{v}_{O'O} = (-7\,\mathrm{m/s})\hat{x}$

d. $r_{OO'} = 21\,\mathrm{m}$

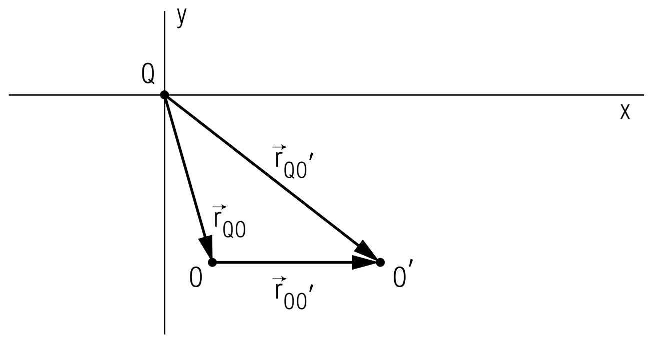
e. $r_{QO} = 91.29\,\mathrm{m}$

f. $r_{QO'} = 102.66\,\mathrm{m}$

g. 10.46° *Help: [S-1]*

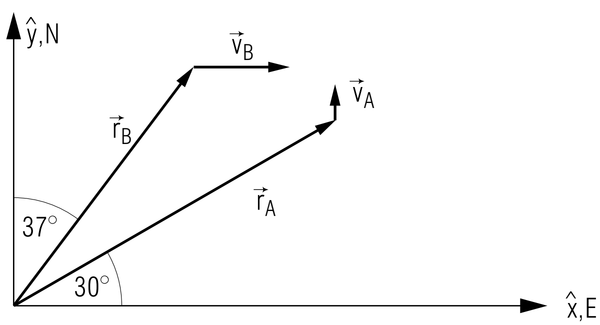
h. 10.46° *Help: [S-2]*

i.



j. In this problem, $v_{O'O} = -7\,\mathrm{m/s}$, so v is not zero at any time.

4. a.



b. $\vec{r}_A = (10.4\,\mathrm{mi})\hat{x} + [6\,\mathrm{mi} + (250\,\mathrm{mi/hr})t]\,\hat{y}$

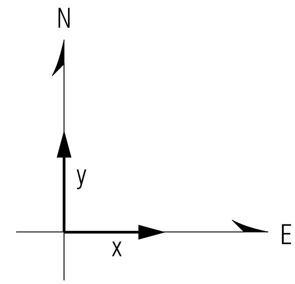
$\vec{r}_B = [6\,\mathrm{mi} + (550\,\mathrm{mi/hr})t]\,\hat{x} + (8\,\mathrm{mi})\hat{y}$

$\vec{r}_{BA} = \vec{r}_B - \vec{r}_A = [(550\,\mathrm{mi/hr})t - 4.4\,\mathrm{mi}]\,\hat{x} + [2\,\mathrm{mi} - (250\,\mathrm{mi/hr})t]\,\hat{y}$

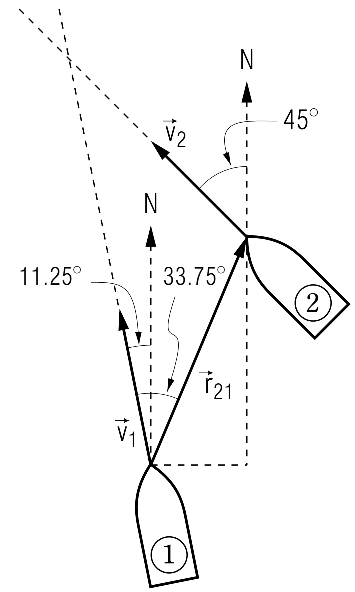
c. The planes will collide in 29 seconds. A computer would be advisable.

d. $\vec{v}_{BA} = 604\,\mathrm{mi/hr}$, bearing 114°.

5. a.



frigate:
$v_1 = 6$ knots
schooner:
$\vec{v}_2 = 8$ knots
$\vec{r}_{21} = 0.25$ mi



b. $\vec{v}_{21} = (-4.48\,\mathrm{knots})\hat{x} + (-0.228\,\mathrm{knots})\hat{y}$

c. $\vec{r}_{21} = [0.096\,\mathrm{mi} + (-4.48\,\mathrm{knots})t]\,\hat{x} + [0.231\,\mathrm{mi} + (-0.228\,\mathrm{knots})t]\,\hat{y}$

d. $r_{21} =$

$$\left([0.096\,\mathrm{mi} + (-4.48\,\mathrm{knots})t]^2 + [0.231\,\mathrm{mi} + (-0.228\,\mathrm{knots})t]^2\right)^{1/2}$$

e. The minimum value of r_{21} is 0.226 miles. This is greater than 0.2 mi, so the schooner will never be in range.

6. 56.6 mi

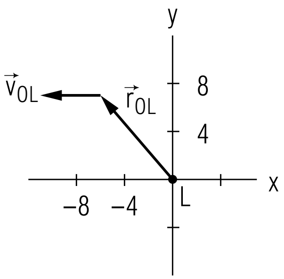
7. a. $\vec{r}_{AL} = (5\,\mathrm{m/hr^2})t^2\hat{x} + (8\,\mathrm{m})\hat{y}$

$\vec{r}_{AO} = (7\,\mathrm{m/hr^2})t^2\hat{x} + (-3\,\mathrm{m\,hr})t^{-1}\hat{z}$

b. $v_{OL} = \left(16\,\mathrm{m^2\,hr^{-4}}t^2 + 9\,\mathrm{m^2\,hr^2}t^{-4}\right)^{1/2}$

c. $a_{OL} = \left(16\,\mathrm{m^2/hr^4} + 36\,\mathrm{m^2\,hr^2}t^{-6}\right)^{1/2}$

d. x, y in meters:



SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-problem 3g)*

Write down the vector relation between $\vec{r}_{QO}$, $\vec{r}_{QO'}$, and $\vec{r}_{O'O}$ and draw a vector addition diagram illustrating this relation. Now use either the law of cosines or the properties of the dot product to find the angle between $\vec{r}_{QO}$ and $\vec{r}_{QO'}$.

S-2 *(from PS-problem 3h)*

Looking along the z-axis, one does not see the z-component of a vector. Then $r^2_{QO} = x^2_{QO} + y^2_{QO}$, etc.

S-3 *(from TX-2e)*

This is a 1-dimensional problem so the vector notation can be dropped:

$$v_{BA} = v_B - v_A\,.$$

You are being observed by the trooper, relative to the trooper, so the equation reads:

$$v_{\text{you relative to trooper}} = v_{\text{you}} - v_{\text{trooper}}\,,$$

and so:

$$v_{\text{you relative to trooper}} = 45\,\text{mph} - 65\,\text{mph} = -20\,\text{mph}\,.$$

S-4 *(from TX-2e)*

Write the equations in [S-3] as vector equations. Align one of the coordinate axes along one of the roads and take components of the equations. There will be sines and cosines in the component equations. After solving the equations, sketch the vectors approximately to scale and see if your answer is approximately right.

S-5 *(from PS, problem 3c)*

Apply the right equations from the text to get: $\vec{r}_{O'O} = -\vec{r}_{QO'} + \vec{r}_{QO}$.

MODEL EXAM

Use double subscripts and proper units wherever possible.

1. See Output Skill K1, this module's ID Sheet.

2. You are a small-craft airplane pilot. You have pointed your plane due west and maintained a speed of 120 mi/hr as registered on your air-speed indicator (speed with respect to the air) for two hours. However, there is a wind blowing out of the north-east at 40 mi/hr. Calculate the distance you are off your due-west course.

3. Relative to the origin of a laboratory reference frame L, an object A's position varies with time according to

$$\hat{r}_A = 5t^2\hat{x} + 8\hat{y}\,,$$

while an observer O sees

$$\hat{r}_A = 7t^2\hat{x} - 3t^{-1}\hat{z}\,.$$

   a. Rewrite the above vector equations, putting in double subscripts and proper units. Assume that distance is measured in meters and time in hours.

   b. Find the speed (magnitude of the velocity) of O relative to L at time t.

   c. Find the magnitude of the acceleration of O relative to L at time t.

   d. Draw a rough sketch of the position and velocity vectors of O, as seen by L at $t = 90$ min, as they would appear looking down the z-axis toward the origin.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 6.

3. See this module's *Problem Supplement*, problem 7.