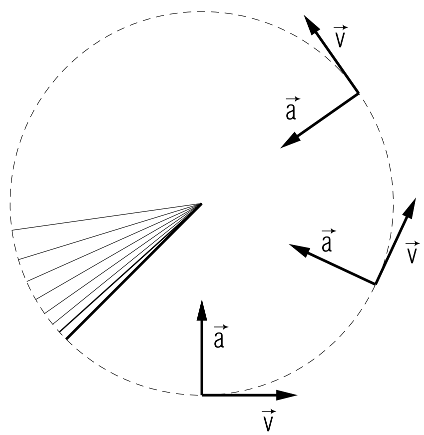
PHYSNET

MISN-0-9

CIRCULAR MOTION: KINEMATICS



$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

CIRCULAR MOTION: KINEMATICS

by

James M. Tanner, Georgia Institute of Technology

Title: **Circular Motion: Kinematics**

Author: James M. Tanner, School of Physics, Georgia Inst. of Tech., Atlanta, GA 30332

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Input Skills:

1. Define kinematically: velocity, speed, and acceleration (MISN-0-7), (MISN-0-8).
2. Use the Cartesian unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ to write a vector in terms of its Cartesian components (MISN-0-1), (MISN-0-2).
3. Use the chain rule to differentiate sine and cosine (MISN-0-1), (MISN-0-2).
4. Given the position, velocity, and acceleration vectors for a particle executing circular motion, determine: (1) if the motion is clockwise or counterclockwise; and (2) if the particle's speed is increasing, decreasing or unchanging (MISN-0-72).
5. Given a particle's position on a circle, its direction of motion as clockwise or counterclockwise, and whether its speed is increasing, decreasing, or unchanging, draw plausible position, velocity, and acceleration vectors with correct spatial orientations (MISN-0-72).

Output Skills (Knowledge):

K1. Define angular velocity in terms of angular displacement and explain how to determine the direction of the angular velocity vector.

Output Skills (Rule Application):

R1. Given a particle's position (as measured along the arc of its circular path) as a function of time, formally calculate its position, velocity, and acceleration vectors at any instant.

R2. A particle in uniform circular motion has these four descriptors: radius, speed, radial acceleration (magnitude), plus either of period or frequency. Given any two of these quantities, determine the other two.

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Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

CIRCULAR MOTION: KINEMATICS

by

James M. Tanner, Georgia Institute of Technology

1. Introduction

1a. Understanding Circular Motion is Important. Circular motion appears routinely in our day-to-day living: amusement park rides, satellites, analog clocks, wheels on cars and bicycles, and rotating driers are just a few of the many examples we encounter. Applications are everywhere, as well, in science and engineering. In addition, circular motion must be studied because it has its own concepts and vocabulary that are widely used.

The special case of circular motion at a constant speed, "uniform circular motion," occurs in so many applications that we give it special attention in this module. The prime example of uniform circular motion which we use here is a spot on the minute hand of an analog clock (the kind of clock that has hands!).

1b. A Special Case of General Motion. Circular motion is just a special case of general motion. Consequently, the concepts and equations treated here are just special cases of concepts and equations of general motion.[1] You can think of this module's topic as a special case that illustrates some interesting and useful facets of those general concepts and equations.

1c. Prerequisites. Below we list some statements assumed to be familiar to the reader of this module from prior study. If any of these statements are not apparent to you, at least upon reflection, you should immediately go back and study the relevant prerequisites[1] before plunging into this module's material. As a particle executes circular motion (see Fig. 1-3): (1) its radius vector, going from the center of the circle to the particle, continually changes direction; (2) its velocity vector continually changes direction to stay tangent to the circle: the velocity points in the direction of motion and has a magnitude (the particle's "speed") equal to |ds/dt| where s measures distance traveled along the particle's circular trajectory; (3) it always has a non-zero radial acceleration, one pointing toward the center of the circle, perpendicular to the velocity vector, that causes the velocity vector to continually change its direction but not its magnitude; (4) it may have a tangential acceleration, one parallel to the velocity vector, that causes the magnitude of the velocity vector to change; (5) the direction of a tangential acceleration is the same as the direction of the velocity if the speed is increasing, and it is opposite to the direction of the velocity if the speed is decreasing; (6) if there is a tangential acceleration, then the (total) acceleration vector points inward but not toward the center of the circle.

[1]See "Trajectories and Radius, Velocity, Acceleration" (MISN-0-72).

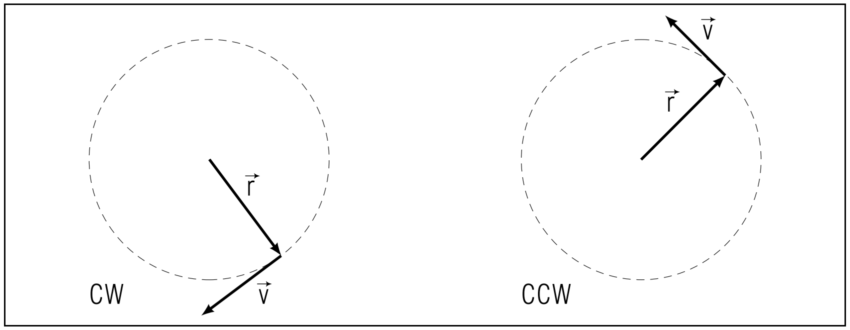

Figure 1. $\vec{v}$ is always tangent to the circle and its direction shows whether the motion is CW (clockwise) or CCW (counterclockwise).

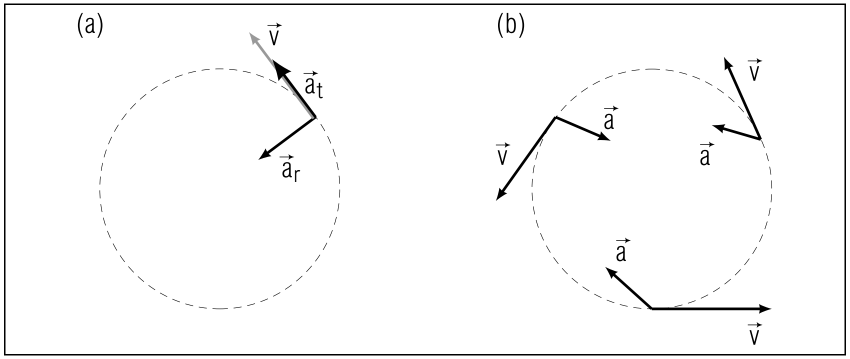

Figure 2. (a.) The radius and velocity vectors change direction as time goes on. (b.) The acceleration always has a radially inward component.

2. Quantitative Kinematics

2a. The General Equations That We Particularize. In this section, for the radius, velocity, and acceleration vectors of circular motion, we formally: (1) look at their directional characteristics; and (2) examine simple equations relating them. We do this by starting with the general kinematical equations relating the radius, velocity, and acceleration vectors,

$$\vec{v} = d\vec{r}/dt \; and \; \vec{a} = d\vec{v}/dt$$

and particularizing them to the circular motion case.

2b. The Radius Vector: Path and Cartesian Coordinates. The position of a particle in circular motion can be specified in terms of its distance s measured along its circular trajectory, starting from the x-axis (see Fig. 3) as the point with $s = 0$. The sign convention for s is important: s is defined to be positive when measured CCW from the x-axis. Thus if s is increasing, so $ds/dt > 0$, the motion is CCW (see Fig. 1). The particle's angle from the x-axis, θ, is given by the ratio of s to r:

$$\theta = \frac{s}{r} \qquad \text{(only for } \theta \text{ in radians!)} \tag{1}$$

Can you write down the more complicated expression for θ in degrees? *Help: [S-9]* Because we like to use simple relationships, most of us think of radians, not degrees, as the natural units for angles in most of physics. We do, however, use degrees for compass bearings.

The Cartesian coordinates of a particle in circular motion are given by the relations:[2]

$$x = r\cos\theta; \qquad y = r\sin\theta\,.$$

Consequently, the particle's position $\vec{r}$ may be written in terms of Cartesian unit vectors as:

$$\vec{r} = x\hat{x} + y\hat{y}\,.$$

2c. The Radius Vector in Polar Coordinates. It is often convenient to express a particle's radius vector in polar coordinates:

$$\vec{r} = r\hat{r}\,. \tag{2}$$

In this system the orthogonal unit vectors are $\vec{r}$ and θ, illustrated in Fig. 4. The unit vector $\hat{\theta}$ always points CCW because its direction is defined as the direction of motion when $\hat{\theta}$ is increasing. The unit vector $\hat{r}$ always

[2]See "... Vectors" (MISN-0-2).

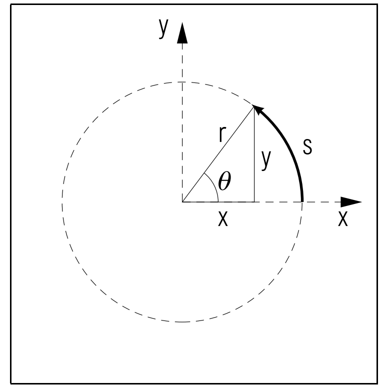

Figure 3. The geometrical relationships between r, s, θ, x, y.

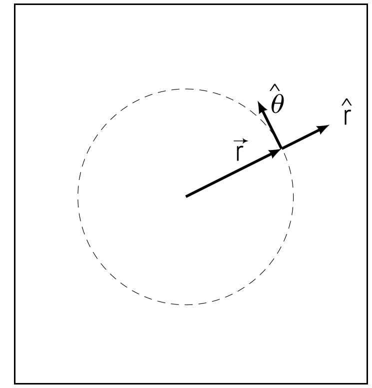

Figure 4. The unit vectors $\hat{r}$, $\hat{\theta}$.

points radially outward (see Fig. 4). Note that the directions of the unit vectors $\hat{r}$ and $\hat{\theta}$ depend on the value of the coordinate θ, in contrast to the Cartesian case where the unit vectors $\hat{x}$ and $\hat{y}$ do not depend on the coordinates x and y.

2d. Velocity and Speed. The velocity of a particle is always tangential to its trajectory, so for circular motion it is tangential to the circle:

$$\vec{v} = (ds/dt)\hat{\theta} \qquad \text{(see Fig. 4)}\,. \tag{3}$$

From Eq. (3) we see that: for CCW motion we have $ds/dt > 0$ and $\vec{v}$ points in the same direction as $\hat{\theta}$; for CW motion, $ds/dt < 0$ and $\vec{v}$ points in the direction opposite to $\hat{\theta}$. Speed v is defined to be the magnitude of the velocity:

$$v = |\vec{v}| = |ds/dt|\,. \tag{4}$$

Thus speed is the (positive) rate at which distance is being traversed around the circle, no matter in which direction:

$$\vec{v} = v\hat{v} = \pm v\hat{\theta}\,.$$

2e. Radial. Circular-motion acceleration must almost always be computed in terms of its tangential and radial components and then those two components must be added vectorially (easy since they are orthogonal). These two components can be derived by differentiating Eq. (3)

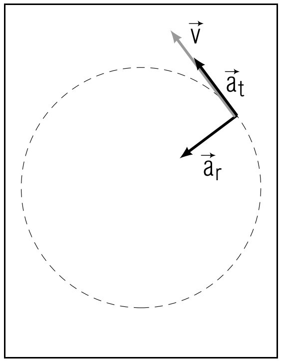

Figure 5. $\vec{a}_r$ for an example of circular motion.

with respect to time (see the Appendix for details), with the result:

$$\vec{a} = d\vec{v}/dt = \vec{a}_t + \vec{a}_r \,, \tag{5}$$

where the tangential and radial vector components are:

$$\vec{a}_t = (dv/dt)\hat{\theta} \,, \tag{6}$$

$$\vec{a}_r = (-v^2/r)\hat{r} \,. \tag{7}$$

A radial component is illustrated in Fig. 5 for a case where $\vec{a}_t$ and $\vec{v}$ are in the same direction so the speed is increasing. Note that $\vec{a}_r$ always points toward the origin in circular motion.

The radial and tangential components of the acceleration are mutually perpendicular so the magnitude of the acceleration is:

$$a = [(dv/dt)^2 + (v^2/r)^2]^{1/2} \,. \tag{8}$$

3. Uniform Circular Motion

3a. Constant Speed. Uniform circular motion is defined to be circular motion in which the particle's speed is constant in time. This condition, usually expressed as $dv/dt = 0$, results in major simplifications to the equations of motion. It also allows us to easily introduce the widely used quantities "period," "frequency," and "angular velocity." Here are some examples of uniform circular motion: (1) a spot on one of the hands of an analog clock; (2) a spot on the turbine of a jet plane traveling at constant speed; (3) a point on the shaft of a motor running at constant speed; and (4) a piece of chewing gum on the tire of a bicycle going at constant speed.

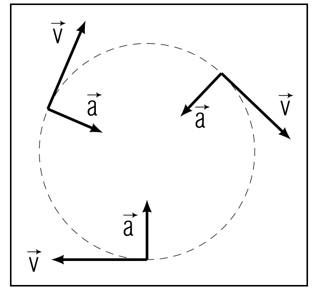

Figure 6. Acceleration in uniform circular motion.

3b. The Acceleration Direction is Radially Inward. Speed v is constant for uniform circular motion, $dv/dt = 0$, and thus the tangential acceleration, whose magnitude is dv/dt, vanishes. The entire acceleration is therefore radial (see Fig. 6) and is given by Eq. (7): $\vec{a} = -(v^2/r)\hat{r}$. Since the acceleration is perpendicular to the velocity, the direction of the velocity is continually changing while the magnitude of the velocity is constant.

4. Parameters of the Motion

4a. The Example We Will Use. Throughout this section we will use the example of an analog clock, a clock with hands. When we describe the motion of a point on the hand, you can think of the point as being half way out from the center to the tip of the hand, or at the tip, or wherever suits you.

4b. Cycle. Basic to the description of uniform circular motion is the idea of the "cycle." This word denotes one complete trip around the circle, from when the particle passes some particular point on the trajectory until it again passes that same point. The descriptor "2.3 cycles" would denote two and three-tenths trips around the circle. A point on the second hand of an analog clock completes 60 cycles for each cycle of a point on the clock's minute hand.

4c. Period. The period T is defined as the time required for each cycle. For example, a point on the minute hand of an analog clock has a period of 60 minutes while the period of the hour hand is 12 hours. The period T of a point is related to the radius r and speed v of the point by:

$$T = \frac{2\pi r}{v} \tag{9}$$

Here the speed was computed as the distance traveled in one cycle (the circumference of the circle) divided by the time for that cycle (the period): $v = 2\pi r/T$. Solving that equation for T gave Eq. (9).

4d. Frequency. The frequency ν (this Greek letter is pronounced "$n\bar{u}$") is defined as the number of cycles divided by the time necessary to complete that number of cycles ("the number of cycles per unit time"). Frequency ν can easily be related to period T by noting that the period is the time for one cycle: $\nu = 1/T$. Using this expression in Eq. 9 gives a second expression for the frequency:

$$\nu = \frac{1}{T} = \frac{v}{2\pi r}\,. \tag{10}$$

For an analog clock, the frequency of a spot on the minute hand is 1 cycle/hr while the frequency of the hour hand is 1 cycle per half day.

The SI (international standard) unit of frequency is the "cycle per second," called the hertz and abbreviated as Hz. Thus if $\nu = 8.5$ Hz, this means that 8.5 cycles are completed each second.

▷ Show that the frequency of a spot on an analog clock's second hand is $\nu = 0.0167$ Hz, to 3 significant digits, while that of the hour hand is $\nu = 0.0000231$ Hz. *Help: [S-17]*

4e. Angular Speed. Angular speed is defined as the time rate of change of angular position θ, and is denoted by the Greek letter ω. In English ω is written "omega" and pronounced "oh-may'-gah." Mathematically,

$$\omega = |d\theta/dt|\,. \tag{11}$$

Using the relation $\theta = s/r$ (where θ is measured in radians) we obtain:

$$\omega = \left|\frac{d}{dt}\left(\frac{s}{r}\right)\right| = \frac{1}{r}\left|\frac{ds}{dt}\right| = \frac{1}{r}v\,. \textit{ Help: [S-16]} \tag{12}$$

Thus the tangential speed, $v = |ds/dt|$, is related to ω by:

$$v = \omega r\,. \tag{13}$$

Note that v, ω, and r are either positive or zero, never negative.

We can derive a very important relation by noting that there are 2π radians per circular cycle, so the rate at which radians are swept out by the radius vector is 2π times the rate at which cycles are completed:

$$\omega = 2\pi(\text{ radians})\nu\,. \tag{14}$$

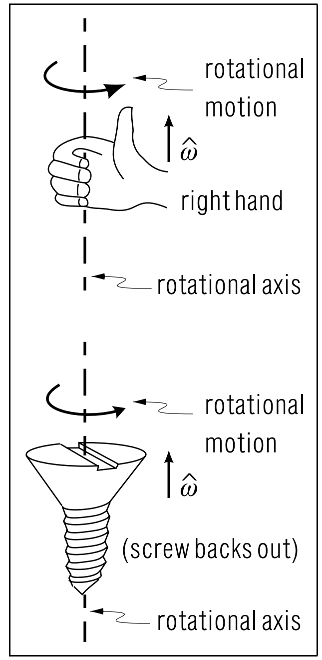

Figure 7. Direction of $\vec{\omega}$ for circular motion: 2 methods.

We normally just write "$\omega = 2\pi\nu$" and leave the "(radians)" as understood.

▷ Show that the angular speed of a point on the minute hand of an analog clock is 0.00175 rad/sec. *Help: [S-18]*

4f. Angular Velocity. Angular velocity is defined as the vector-like quantity $\vec{\omega}$ whose magnitude is defined to be the angular speed, ω, and whose direction is defined to be: *Help: [S-8]*

$$\hat{\omega} = \hat{r} \times \hat{v}\,. \qquad \text{(circular motion only)} \tag{15}$$

The direction $\hat{\omega}$ always lies along the "axis of rotation." There are two ways of determining the direction in which $\hat{\omega}$ points along the rotation axis (see Fig. 7). Pick the method you like best:

First Way: Wrap the fingers of your right hand around the axis, pointing them in the direction of rotational motion: your thumb then points in the direction of $\hat{\omega}$.

Second Way: The direction of $\hat{\omega}$ is the direction in which a right hand screw thread (the standard thread) would advance if moved in the same

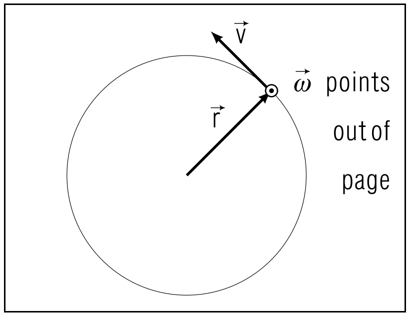

Figure 8. $\vec{\omega}$, $\vec{r}$ and $\vec{v}$.

direction as the motion (imagine opening or closing a jar lid). For a point moving with angular velocity $\vec{\omega}$ in a circle whose position is $\vec{r}$ (See Fig. 8), the point's tangential velocity is: $\vec{v} = \vec{\omega} \times \vec{r}$.

Note that $\hat{\omega} \neq d\hat{\theta}/dt$ because $\hat{\omega}$ and $d\hat{\theta}/dt$ are mutually perpendicular (make sure you see that!).

Mathematically, $\hat{\omega}$ is a "pseudo-vector" because it does not behave the same way as a true vector under reflection in a mirror. Its other properties are those of true vectors such as $\vec{r}$, $\vec{v}$, and $\vec{a}$.

5. Why Forces are Not Considered

5a. The Problem. A number of people encountering the subject of kinematics for the first time have asked questions like this: "You calculated the acceleration of the pilot in the bottom of the dive, using the plane's speed and radius of curvature, but you didn't take into account the force of gravity. How come?" Suppose a friend studying physics asked you that question. How would you reply?

5b. The Answer. Forces are never considered in a kinematics problem. There is no symbol representing a force in any of the kinematics equations. The reason that forces do not come into kinematics calculations is that kinematics is complete without knowledge of the forces involved. Put another way, the same information that forces would supply is already completely given to you in the statement of a kinematics problem.

For the example given above, forces determine the plane's speed and radius of curvature and those quantities are then used to calculate the acceleration. However, in a kinematics problem you are given the speed

and radius of curvature in the statement of the problem; therefore you do not need to calculate them using forces. You can just go ahead and calculate the acceleration.

Acknowledgments

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A. Derivations: for Those Interested

Show: $\vec{v} \perp \vec{r}$. First we write down the unit vector in the radial direction:

$$\hat{r} = \hat{x}\cos\frac{s}{r} + \hat{y}\sin\frac{s}{r}\,. \tag{16}$$

> Exercise 1: Show that: (a) $\hat{r}$ is a unit vector; and (b) $\hat{r}$ points in the same direction as $\vec{r}$. *Help: [S-1]*

Differentiating Eqs. (2) and (16) with respect to time gives the velocity:[3] *Help: [S-4]*

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = \frac{d(r\hat{r})}{dt} = r\frac{d\hat{r}}{dt} = r\frac{d\hat{r}}{ds}\frac{ds}{dt}\,,\\ &= \frac{ds}{dt}\left(-\hat{x}\sin\frac{s}{r} + \hat{y}\cos\frac{s}{r}\right)\,.\end{aligned} \tag{17}$$

We can write this as:

$$\vec{v} = \frac{ds}{dt}\hat{\theta}\,,$$

where

$$\hat{\theta} = -\hat{x}\sin\frac{s}{r} + \hat{y}\cos\frac{s}{r}\,. \tag{18}$$

> Exercise 2: Show that: (a) $\hat{\theta}$ is a unit vector; (b) $\hat{\theta}\cdot\vec{r} = 0$; and (c) $\hat{\theta}$ is perpendicular to $\vec{r}$. *Help: [S-2]*

[3] See "Review of Mathematical Skills-Calculus: Differentiation and Integration" (MISN-0-1).

Derive: tangential and radial acceleration. Start with the most general relation between velocity and acceleration and particularize it to the case of circular motion:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\hat{\theta}\right) = \frac{d^2s}{dt^2}\hat{\theta} + \frac{ds}{dt}\frac{d\hat{\theta}}{dt}\,.$$

Differentiating Eqs. (1) and (18):

$$\frac{d\hat{\theta}}{dt} = -\frac{1}{r}\frac{ds}{dt}\hat{r}\,.$$

Then:

$$\vec{a} = \left(\frac{d^2s}{dt^2}\right)\hat{\theta} - \frac{1}{r}\left(\frac{ds}{dt}\right)^2\hat{r}\,. \qquad (19)$$

Since $\hat{\theta}$ is tangent to the circle, the tangential component of the acceleration is:

$$\vec{a}_t = (d^2s/dt^2)\hat{\theta}\,,$$

and its magnitude is

$$a_t = |\vec{a}_t| = |d^2s/dt^2| = |dv/dt|\,.$$

Exercise 3: Show that if $\vec{a}_t$ and $\vec{v}$ in the same direction then the speed is increasing; and that if $\vec{a}_t$ and $\vec{v}$ are in opposing directions then the speed is decreasing. *Help: [S-5]*

The second term of Eq. (19) is the radial component of $\vec{a}$:

$$\vec{a}_r = -\frac{1}{r}(ds/dt)^2\hat{r}\,.$$

Since $(1/r)\cdot(ds/dt)^2$ is always positive, $\vec{a}_r$ is opposite to $\vec{r}$ and points radially inward. Squaring $v = |ds/dt|$ we get $v^2 = (ds/dt)^2$ so the radial component of the acceleration has magnitude $a_r = v^2/r$ and direction $(-\hat{r})$. Consequently the radial acceleration is:

$$\vec{a}_r = -\frac{v^2}{r}\hat{r}\,.$$

Acceleration and Velocity are Perpendicular in UCM. For UCM, $d^2s/dt^2 = 0$ so:

$$\vec{a} = \vec{a}_r + \vec{a}_t = -a_r\hat{r}\,,$$

$$\vec{v} = (ds/dt)\hat{\theta}\,.$$

Now $\hat{r}$ and $\hat{\theta}$ are mutually perpendicular so $\vec{a}$ and $\vec{v}$ are mutually perpendicular.

PROBLEM SUPPLEMENT

Exercises

1. Show that, for uniform circular motion: $a_r = 4\pi^2 r/T^2 = 4\pi^2\nu^2 r$. *Help: [S-6]*

2. Show that, for uniform circular motion:

   a. $\omega = 2\pi\nu$

   b. $a_r = \omega^2 r$ *Help: [S-7]*

3. a. Explain why $v = |ds/dt|$, $v \neq ds/dt$.

   b. If s is graphed versus time, how can v be determined from this graph? *Help: [S-3]*

Problems

Note: Remember to set your calculator to the proper angular measure, radians or degrees, for the equations you are using! *Help: [S-12]*

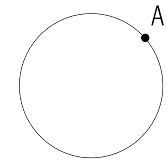
1. The moon orbits the earth in an approximately circular path (mean radius $= 3.84 \times 10^5$ km) every 27.3 days. Calculate the moon's speed and radial acceleration in this orbit. *Help: [S-25]*

2. Calculate the speed and acceleration of a person on the equator resulting from the earth's spinning about its axis through the poles. The earth's mean radius is 6.37×10^6 m. *Help: [S-10]*

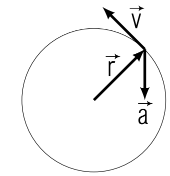
3. At a particular instant of time a car on a circular track (radius $= 8.0 \times 10^2$ ft) is traveling at 50.0 miles per hour and is increasing its speed at a rate of 5.0 miles per hour per second. Determine the magnitude of its acceleration, at this particular instant of time, in ft/s^2. *Help: [S-11]*

4. Calculate the acceleration of a pilot at the bottom of a dive in a plane traveling at a constant 5.0×10^2 miles per hour along a circular path of radius 2.0 miles.

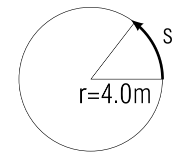
5. The particle at point A is traveling clockwise with an increasing speed. Draw its velocity and acceleration vectors.



6. The position, velocity, and acceleration vectors for a particle executing circular motion are shown. Is the particle traveling CW (clockwise) or CCW (counterclockwise)? Is its speed increasing, decreasing, or constant?



7. *Note: If you have trouble with any part of this problem: (1) make sure you have thoroughly understood each and every paragraph of this module's text; (2) work all the other problems successfully before tackling this one; (3) if necessary, review the prerequisite material on derivatives in MISN-0-1 and unit vectors in MISN-0-2; and (4) use the references, labeled Help: [S-], in this module's Special Assistance Supplement.*



An object moves around the circle shown such that:

$$s(t) = (8.0\,\text{m/s})t - (1.00\,\text{m/s}^2)t^2\,.$$

The problem is to determine the object's kinematic variables at several different times.

a. To start out with, write: $s(t) = bt + ct^2$.
   Now determine $\vec{r}$ in terms of r, b, c, and t. *Help: [S-15]*

b. Determine $\vec{v}$ and v in terms of r, b, c, and t. *Help: [S-19]*

c. Determine $\vec{a}$ and a in terms of r, b, c, and t. *Help: [S-20]*

d. Substitute the numerical values for r, b, and c and determine the object's (numerical) position, in terms of Cartesian unit vectors, at $t = 2.0$ sec, 4.0 sec, and 5.0 sec. *Help: [S-21]*

e. Determine the velocity, in terms of Cartesian unit vectors, and the speed at $t = 2.0$ sec, 4.0 sec, and 5.0 sec. *Help: [S-22]*

f. Determine the acceleration, in terms of Cartesian unit vectors, and the magnitude of acceleration at $t = 2.0$ sec, 4.0 sec, and 5.0 sec. *Help: [S-23]*

g. Determine v from ds/dt and a from d^2s/dt^2 and v (by computing a_t and a_r), for $t = 2.0\,\text{sec}$. *Help: [S-24]* Compare to the answers obtained in parts (e) and (f) above.

h. Sketch $\vec{r}$, $\vec{v}$, and $\vec{a}$ for each of the times of part (d).

i. Determine whether the speed is increasing, decreasing, or not changing at each of the times of part (d).

j. What is happening to the direction of motion at $t = 4\,\text{s}$?

Brief Answers:

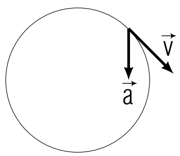
1. $1.02 \times 10^3\,\text{m/s}$, $2.72 \times 10^{-3}\,\text{m/s}^2$ inward.

2. $463\,\text{m/s}$, $0.0337\,\text{m/s}^2$ (toward center of earth).

3. $9.9\,\text{ft/s}^2$.

4. $51\,\text{ft/s}^2$ (toward center of circle).

5.



6. CCW, decreasing.

7. a. $\vec{r} = r\cos\theta\hat{x} + r\sin\theta\hat{y}$, where: $\theta = (bt + ct^2)/r$.

b. $\vec{v} = (b + 2ct)\cdot(-\sin\theta)\hat{x} + (b + 2ct)\cdot(+\cos\theta)\hat{y})$

c. $\vec{a} = [-2c\sin\theta - (b + 2ct)^2(1/r)\cos\theta]\hat{x} + [+2c\cos\theta - (b + 2ct)^2(1/r)\sin\theta]\hat{y}$

d. $\vec{r}(2.0\,\text{s}) = (-4.0\hat{x} + 0.6\hat{y})\,\text{m}$.

$\vec{r}(4.0\,\text{s}) = (-2.6\hat{x} - 3.0\hat{y})\,\text{m}$.

$\vec{r}(5.0\,\text{s}) = (-3.3\hat{x} - 2.3\hat{y})\,\text{m}$.

e. $\vec{v}(2.0\,\text{s}) = (-0.6\hat{x} - 4.0\hat{y})\,\text{m/s}$,v$= 4.0\,\text{m/s}\,(CCW)$.

$\vec{v}(4.0\,\text{s}) = (+0.0\hat{x} - 0.0\hat{y})\,\text{m/s}$, $v = 0.0\,\text{m/s}$ (neither).

$\vec{v}(5.0\,\text{s}) = (-1.1\hat{x} + 1.6\hat{y})\,\text{m/s}$, $v = 2.0\,\text{m/s}$ (CW).

f. $\vec{a}(2.0\,\text{s}) = (+4.2\hat{x} + 1.4\hat{y})\,\text{m/s}^2$, $a = 4.5\,\text{m/s}^2$.

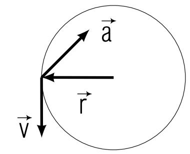
$\vec{a}(4.0\,\text{s}) = (-1.5\hat{x} + 1.3\hat{y})\,\text{m/s}^2$, $a = 2.0\,\text{m/s}^2$.

$\vec{a}(5.0\,\text{s}) = (-0.3\hat{x} + 2.2\hat{y})\,\text{m/s}^2$, $a = 2.2\,\text{m/s}^2$.

g. $v(2.0\,\text{s}) = \dfrac{ds}{dt}(2.0\,\text{s}) = 4.0\,\text{m/s}$

$a(2.0\,\text{s}) = \sqrt{a_r^2 + a_t^2} = 4.5\,\text{m/s}^2$.

h. The sketch is for $t = 2$ sec. For the other times, check that the vectors agree with the values you computed above and with common sense.



i. For 2, 4, 5 sec: decreasing, not changing, increasing.

j. It is reversing.

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-Appendix Exercise 1)*

a. $\hat{r}\cdot\hat{r} = \left|\dfrac{\vec{r}}{r}\right|\left|\dfrac{\vec{r}}{r}\right|(\cos\theta) = \dfrac{|\vec{r}|^2}{r^2}(1) = 1$, so $\hat{r}$ is a unit vector.

b. $\hat{r}$ and $\vec{r}$ are in the same direction so $\theta = 0$ so $\hat{r}\cdot\vec{r} = r$.

S-2 *(from TX-Appendix Exercise 2)*

a. See if $\hat{\theta}\cdot\hat{\theta} = 1$ as for $\hat{r}$ in [S-1].

b. $\hat{\theta}\cdot\hat{r} = 0$, so either: (1) $|\hat{\theta}| = 0$; or (2) $|\hat{r}| = 0$; or (3) they are perpendicular. Choices (1) and (2) are obviously false so (3) must be true.

S-3 *(from PS-Exercise 3)*

a. $|ds/dt|$ and ds/dt could differ in sign. What is the sign of the magnitude of a vector?

b. The derivative is the slope of the graph.

S-4 *(from TX-Appendix, Sect. 1)*

Use the chain rule:

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{ds}\frac{ds}{dt}$$

S-5 *(from TX-Appendix Exercise 3)*

You are given $\vec{v}$ and are asked about $\vec{v}+\Delta\vec{v}$ a short time Δt later. $\Delta\vec{v}$ is $\vec{a}_t\Delta t$. Draw vector addition diagrams to decide if $\vec{v}+\Delta\vec{v}$ is longer than, shorter than, or the same length as $\vec{v}$.

S-6 *(from PS-Exercise 1)*

Use $a_r = v^2/r$ and $v = 2\pi r/T$.

S-7 *(from PS-Exercise 2)*

a. Use $v = \omega\, r$ and $\nu = v/2\pi r$.

b. Use $a = v^2/r$ and $v = \omega\, r$.

S-8 *(from TX-4f)*

The direction of $\vec{\omega}\times\vec{r}$, by definition of the vector product, must be perpendicular to $\vec{\omega}$. Therefore this vector product lies in the plane of motion. It must also be perpendicular to $\vec{r}$; therefore it lies along the tangent. Using the right hand rule, the product points in the direction of the tangential velocity. Since $\vec{\omega}$ and $\vec{r}$ are perpendicular, the magnitude of $\vec{\omega}\times\vec{r}$ is just $\omega\, r$. This, in turn, is just the magnitude of the tangential velocity, v.

S-9 *(from TX-2b)*

$\theta/180° = s/(\pi r)$

S-10 *(from PS-Problem 2)*

The Earth makes one complete rotation per day: that's what a day is! Therefore the Earth's period of rotation T is ... (you finish it). Draw a diagram and put the vectors on it.

S-11 *(from PS-Problem 3)*

The car's speed is 50 mph only at the quoted instant of time. Slightly earlier in time its speed was less than that, and slightly later its speed will be greater. Therefore it has tangential acceleration. Because it is in circular motion, it also has radial acceleration. *Help: [S-13]*

S-12 *(from TX-2a)*

For help on whether you should use radians or degrees, read the text of this module.

S-13 *(from [S-11])*

The tangential acceleration is 5.0 miles per hour per second, which is $7.33\,\mathrm{ft/s^2}$ (you must check that conversion from miles to feet and from hour to seconds to make sure you understand what is going on). *Help: [S-14]*

S-14 *(from [S-13])*

You must compute $\vec{a}_r$ and $\vec{a}_t$ and add them as vectors. Note that they are mutually perpendicular vectors.

S-15 *(from PS-Problem 7a)*

To get $\vec{r}$ from $s(t)$, see this module's *Table of Contents* and read the sections titled "Radius vector . . . "
It is a good exercise to write down the general expression for $r(\vec{t})$ in terms of r, $\hat{x}$, $\hat{y}$, and any arbitrary $s(t)$. Then substitute in this problem's functional form for $s(t)$.

S-16 *(from TX-4e)*

Note that r is constant for circular motion and use the chain rule for differentiation of products (see MISN-0-1).

S-17 *(from TX-4d)*

The "second" hand makes one complete circuit of the dial in one minute, which is 60 seconds, so its period is: $T = 60\,\mathrm{sec}$. Now convert this to frequency.
The "hour" hand makes one complete circuit of the dial in 12 hours, which is $12 \times 3600\,\mathrm{sec}$, so its period is: $T = 43,200\,\mathrm{sec}$. Now convert this to frequency.

S-18 *(from TX-4e)*

See Table of Contents, "Angular Speed."

S-19 *(from PS-Problem 7b)*

This just involves simple differentiation, such as:

$$\frac{d\cos\theta}{dt} = \frac{d\cos\theta}{d\theta}\cdot\frac{d\theta}{dt} = -\sin\theta\,\frac{d}{dt}\,\frac{bt+ct^2}{r} = -\sin\theta\,\frac{b+2ct}{r}$$

Help: [S-26]

S-20 *(from PS-Problem 7c)*

This involves the same technique as used in [S-19] for $\vec{v}$, except that one must also use the chain rule for differentiation:

$$\frac{d}{dt}\vec{v} = \frac{d}{dt}[-(b+2ct)(\sin\theta)\ldots = -\sin\theta\cdot\frac{d(b+2ct)}{dt} - (b+2ct)\frac{d\sin\theta}{dt}\ldots$$

Help: [S-26]

S-21 *(from PS-Problem 7d)*

Was your calculator (properly) set on radians? Did you remember to divide s by r to get θ?

S-22 *(from PS-Problem 7e)*

$$v = \sqrt{v_x^2 + v_y^2}$$

S-23 *(from PS-Problem 7f)*

$$a = \sqrt{a_x^2 + a_y^2}$$

S-24 *(from PS-Problem 7g)*

See the *Table of Contents*, "Velocity and Speed," and "Acceleration: Tangential and Radial."

S-25 *(from PS-Problem 1)*

Do all the Exercises (above the Problems) successfully before tackling this problem.

S-26 *(from [S-19] and [S-20])*

Write down the general expression for $\vec{r}(t)$ in terms of r, $\hat{x}$, $\hat{y}$, and any arbitrary $s(t)$. Then make sure you can obtain these general expressions from it:

$$d\vec{r}/dt = -\sin(s/r)(ds/dt)\hat{x} + \cos(s/r)(ds/dt)\hat{y}$$

$$\begin{aligned} d^2\vec{r}/dt^2 = & \quad [-\cos(s/r)(ds/dt)^2 - r\sin(s/r)d^2s/dt^2][\hat{x}/r] \\ & + \quad [-\sin(s/r)(ds/dt)^2 + r\cos(s/r)d^2s/dt^2][\hat{y}/r] \end{aligned}$$

MODEL EXAM

1. See Output Skill K1 in this module's *ID Sheet*, just inside the cover.

2. A particle moves CCW on a circle of radius $r = 4.0\,\mathrm{m}$. Its trajectory position s along the circular arc is given by:

$$s(t) = (8.0\,\mathrm{m/s})t - (1.00\,\mathrm{m/s^2})t^2\,.$$

   a. Determine the particle's position $\vec{r}$, velocity $\vec{v}$, and acceleration $\vec{a}$ at $t = 2.0\,\mathrm{s}$, in terms of the Cartesian unit vectors $\hat{x}$ and $\hat{y}$. Sketch $\vec{r}$, $\vec{v}$, and $\vec{a}$ at this time.

   b. Determine if the motion of the particle is clockwise or counterclockwise at $t = 2.0\,\mathrm{s}$, and whether the particle's speed is increasing or decreasing at that time.

3. Calculate the speed and acceleration of a person on the equator resulting from the earth's spinning about its axis through the poles. The earth's mean radius is $6.37 \times 10^6\,\mathrm{m}$.

Brief Answers:

1. See this module's *text*.

2. See this module's *Problem Supplement*, problem 7.

3. See this module's *Problem Supplement*, problem 2.