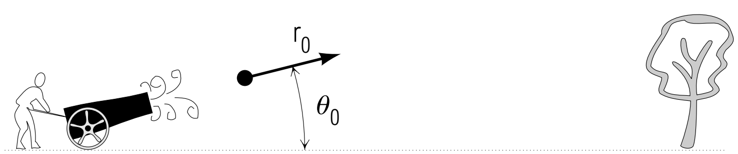
PHYSNET

MISN-0-8

TWO-DIMENSIONAL MOTION



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TWO-DIMENSIONAL MOTION

by

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Title: **Two-Dimensional Motion**

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Input Skills:

1. Vocabulary: displacement, velocity and acceleration vectors (MISN-0-7).
2. Differentiate an expression which includes unit vectors $\hat{x}$ and $\hat{y}$. (MISN-0-2).
3. Find velocity, given the time-dependent position of an object (MISN-0-7).
4. Find acceleration, given the time-dependent velocity. (MISN-0-7).
5. Find position given initial conditions and either velocity or acceleration in one dimension (MISN-0-7).
6. Write the specific equations describing motion in one dimension with constant acceleration (MISN-0-7).

Output Skills (Knowledge):

K1. Write the vector equations relating position, velocity and acceleration in component form.

K2. Explain how the motion of component vectors can be used to describe the motion of an object.

Output Skills (Problem Solving):

S1. Given (either graphically or analytically) one of the functions $\vec{r}(t)$, $\vec{v}(t)$, or $\vec{a}(t)$, plus initial conditions, find the two other functions.

S2. Given $\vec{r}(t)$, derive the equation of the trajectory for an object.

S3. Given a special case of constant velocity or constant acceleration for one component, write the appropriate equations of motion by using pre-derived one dimensional relationships.

S4. Determine the range, maximum height and equation of the trajectory for an object in ballistic motion.

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1. Introduction

1a. Why We Study Motion in Two Dimensions. The real world is three-dimensional, so why do we bother with two-dimensional motion? First, two-dimensional motion is easier to describe, easier to deal with mathematically, and easier to sketch on a piece of flat paper. This makes two-dimensional motion a good place for introducing concepts that are peculiar to motion in more than one dimension. Second, many objects actually do exhibit motion in a plane, motion that needs only two dimensions for its complete description. Any motion under constant acceleration can always be described in terms of just two dimensions. Even if the acceleration is not constant, many objects still move in a plane (e.g., a tractor on a level field, a rider on a ferris wheel).

1b. The Job at Hand. Our basic kinematical problem is to give quantitative information about the time-dependent positions, velocities, and accelerations of objects. This information is to be specified either formally ("analytically"), in words, or graphically.

1c. The Fundamental Relationship. Here are the equations that summarize the fundamental relationships used in this module:[1]

$$\text{position } = \vec{r} \tag{1}$$

$$\text{displacement } = \Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \tag{2}$$

$$\text{average velocity } = \vec{v}_{av} = \Delta\vec{r}/\Delta t \tag{3}$$

$$\text{inst. vel. } = \vec{v} = d\vec{r}/dt \tag{4}$$

$$\text{av. accel. } = \vec{a}_{av} = \Delta\vec{v}/\Delta t \tag{5}$$

$$\text{inst. accel. } = \vec{a} = d\vec{v}/dt = d^2\vec{r}/dt^2 \tag{6}$$

Equation (4) can be inverted, giving:

$$\vec{r}(t) = \int_0^t \vec{v}(t')\,dt' + \vec{r}(0). \tag{7}$$

[1] See also "Kinematics in Three Dimensions" (MISN-0-37).

Equation (6) can be inverted, giving:

$$\vec{v}(t) = \int_0^t \vec{a}(t')\,dt' + \vec{v}(0). \tag{8}$$

2. Analysis by Components

2a. Vector Equations Reduce to Component Equations. Equationstoref18 are vector equations. Each equation could be rewritten as two x and y component equations, so that the two dimensional motion of the object could also be treated as two simultaneous one-dimensional problems. For instance, Eq. (3) is equivalent to the two one-dimensional equations $v_{x,av} = \Delta x/\Delta t$ and $v_{y,av} = \Delta y/\Delta t$. Equation (4) is equivalent to the two one-dimensional equations $v_x = dx/dt$ and $v_y = dy/dt$.

2b. Component Descriptors Relate to Actual Motion. The x and y position, in terms of components, displacement, velocity and acceleration vectors can be related more graphically to the actual motion of the object. The component description is more than a mere exercise in mathematical symbolism.

2c. Motion of the *x*- and *y*-Component Vectors. As a particle moves along a complicated path, as in Fig. 1, its position vector $\vec{r}$ and the component vectors $\vec{x}$ and $\vec{y}$ also move. Envision in your mind how each component vector tip moves as the particle moves from A to B.

2d. Describing the Component Motion. The one-dimensional motion of the tip of the $\vec{x}$ vector can be described[2] by its position x, velocity v_x, and acceleration a_x. Similarly, the motion of the $\vec{y}$ vector can be described by y, v_y and a_y. You can thus use two one-dimensional motions to completely describe one two-dimensional motion.

2e. Dual Roles of Component Descriptors. The quantities x, y, v_x, v_y, a_x, and a_y can be used in two different ways. They can either describe the motion of the tips of the $\vec{x}$ and $\vec{y}$ vectors, or they can describe the components of the actual displacement, velocity and acceleration vectors of the moving object. Problem-solving techniques can take advantage of this dual meaning.

[2] See "Kinematics in One Dimension" (MISN-0-7).

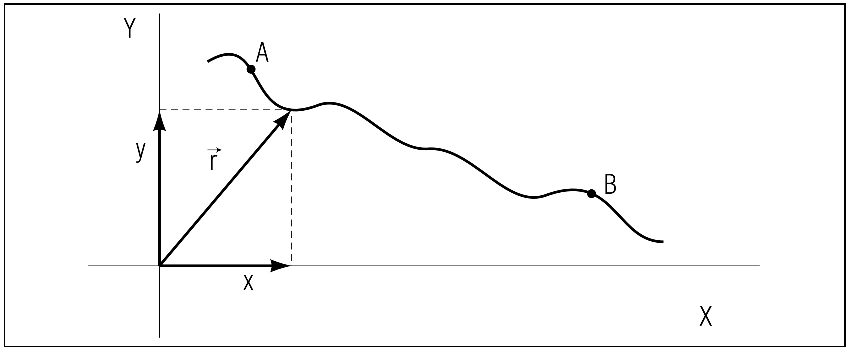


Figure 1. A particle moving on a two dimensional path has x and y component vectors that each move in one dimension.

3. Problem-Solving Techniques

3a. Methods of Specifying Information. Typically, information about the motion of an object is specified in either analytical or graphical form. The position, velocity or acceleration, or some combination of their components, might be given. We can detail some differences between the handling of analytical vs. graphical data.

3b. Analytical Method: General Approach. Equations (1)-(8) analytically relate the variables $\vec{a}$, $\vec{v}$, and $\vec{r}$. If an analytical form for $\vec{a}(t)$, $\vec{v}(t)$ or $\vec{r}(t)$ can be found, the derivatives or integrals can be performed so that $\vec{a}$, $\vec{v}$ and $\vec{r}$ will all be known. The path or "trajectory" of the object can be found in the form of an equation for $y(x)$ by eliminating t between the equations for $x(t)$ and $y(t)$. The trajectory could also be found by plotting (x, y) points for appropriate time values and then drawing a smooth curve through those points.

3c. Two Special Cases. Often, an object moves in such a way that the x- and/or y-component vector tip moves with either constant velocity or constant acceleration. In these cases x, v_x, a_x (and/or y, v_y, a_y) are related by the previously derived equations for an object with one dimensional constant acceleration. The need for differentiation and integration is bypassed.

3d. Graphical Method. Sometimes the time dependence for the components of one of the quantities $\vec{a}(t)$, $\vec{v}(t)$ or $\vec{r}(t)$ is given in graphical form. The interpretation of the derivative as the (physical) slope of a curve and integral as the (physical) area under a curve could then be used to find $\vec{a}$, $\vec{v}$ and $\vec{r}$.

4. Examples and Cautions

4a. Sample Problem: Analytical Method. A sample problem will illustrate the techniques of the analytical method. Suppose that you are given that the velocity of an object is a constant 2 m/s in the x-direction, and that it increases linearly with t in the y-direction: $v_y = (3\,\mathrm{m/s^2})\,t$. Then:

$$\vec{v}(t) = (2\,\mathrm{m/s})\hat{x} + (3\,\mathrm{m/s^2})\,t\hat{y}.$$

The acceleration is:

$$\vec{a} = \frac{d\vec{v}}{dt} = 0\hat{x} + 3\,\mathrm{m/s^2}\,\hat{y}.$$

That is, the object has a constant acceleration in the y-direction. The position of the particle is:

$$\begin{aligned}\vec{r}(t) &= \int_0^t \vec{v}(t')\,dt' + \vec{r}(0)\\ &= \int_0^t (2\,\mathrm{m/s}\,\hat{x} + 3\,\mathrm{m/s^2}\,t'\,\hat{y})\,dt' + \vec{r}(0)\\ &= (2\,\mathrm{m/s})\,t\,\hat{x} + \frac{1}{2}(3\,\mathrm{m/s^2})t^2\hat{y} + \vec{r}(0).\end{aligned}$$

If the problem further stated $\vec{r}(0)$; for example, as "initially the object is at $x = 4\,\mathrm{m}$, $y = 5\,\mathrm{m}$," we could write:

$$\vec{r}(t) = (2\,\mathrm{m/s}\,t + 4\,\mathrm{m})\hat{x} + \left(\frac{3}{2}\,\mathrm{m/s^2}\,t^2 + 5\,\mathrm{m}\right)\hat{y}.$$

The path can now be found. The above vector equation gives the component equations: $x(t) = (2\,\mathrm{m/s})\,t + 4\,\mathrm{m}$; $y(t) = [(3/2)\,\mathrm{m/s^2}]\,t^2 + 5\,\mathrm{m}$. Solving $x(t)$ for t and substituting that into $y(t)$ gives $y(x)$:

$$y(x) = \frac{3}{2}\,\mathrm{m/s^2}\left(\frac{x - 4\,\mathrm{m}}{2\,\mathrm{m/s}}\right)^2 + 5\,\mathrm{m},$$

which is the equation of a parabola: the object moves in a parabolic path.

Note for those interested. The integration could have been bypassed by noting that the x-component motion is at constant velocity and the y-component motion is at constant acceleration. Thus $x(t)$ and $y(t)$ fit the general form of the one-dimensional constant acceleration equation: $x = x_0 + v_0 t + at^2/2$. For example, for the y-direction: $a = 3\,\mathrm{m/s^2}$, $v_0 = 0$, $x_0 \equiv y_0 = 5\,\mathrm{m}$, so: $y = 5\,\mathrm{m} + 0 + (3\,\mathrm{m/s^2})\,t^2/2$.

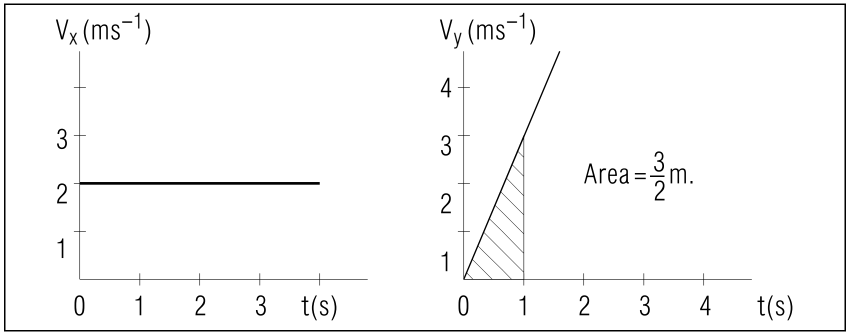


Figure 2. The relations $v_x(t)$ and $v_y(t)$ specified graphically.

4b. Sample Problem: Graphical Method. The previous problem could have specified the velocity components graphically, as in Fig. 2. Now a_x is the slope of the tangent line to the v_x(t) curve: in this case, Fig. 2, the slope is always zero. Similarly, a_y is the slope of the v_y(t) curve, which in this case is always $3\,\mathrm{m/s}^2$.

The x and y coordinates can be found using the area under the curve. For example, to calculate $y(1\,\mathrm{sec})$, Eq. (7) gives:

$$y(1\,\mathrm{s}) = \int_0^{1\,\mathrm{s}} v_y(t')\,dt' + y(0).$$

The integral is given by the shaded area[3] in Fig. 2, so that:

$$y(1\,\mathrm{s}) = \frac{3}{2}\,\mathrm{m} + 5\,\mathrm{m} = \frac{13}{2}\,\mathrm{m},$$

which can be verified from the previous analytical solution for $y(t)$.

4c. The Artificial Nature of the Examples. The real world does not usually present motion problems so neatly specified as the previous examples. These examples have presented information as given that in actuality must have been derived from other information. For example, knowledge of applied forces gives information about the acceleration.[4] Also, coordinate systems and initial times have been implicitly chosen.

[3]This area must be calculated as the "physical" area, not the geometric area. See "The Counting Squares Technique for Numerical Integration" (MISN-0-250, Appendix A).

[4]See "Particle Dynamics" (MISN-0-14).

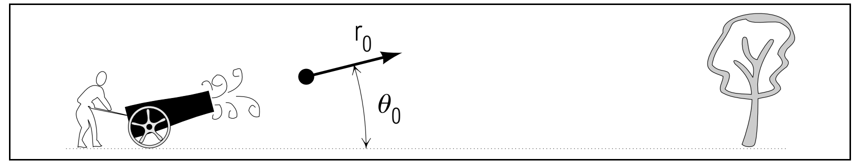


Figure 3. A cannon fires a projectile at an angle θ above the horizontal.

4d. Choice of Coordinates. If the coordinate system is unspecified in a problem, you may choose to use any system you desire. The motion of the object will not depend on the coordinate system that you use to describe the motion. Be prepared to try different coordinate systems; the "best" choice will ease the mathematical manipulation in the problem.

5. Ballisitic Motion

5a. Falling and Free Falling. The acceleration of an object falling above the Earth depends upon its distance from the Earth's surface and upon air resistance. You are familiar with this motion, for example, when you observe a baseball in flight. If the speed of the object is sufficiently low, the effects of the air resistance are negligible.[5] If the object's path does not vary significantly in altitude, the effects of gravity are constant. Under these special conditions, called ballistic motion, the object is "free falling" and will have a constant acceleration of $g = 9.8\,\mathrm{m/s}^2$ vertically downward, and will therefore move in a plane.

5b. Ballistic Motion Example. A projectile is fired with initial velocity $\vec{v}_0$ making an angle θ with the horizontal (see Fig. 3). Ignore the height of the end of the barrel.

We choose a coordinate system such that the horizontal coordinate is x, the origin is at the cannon (so $\vec{r}_0 \equiv \vec{r}_0 = 0$), and the vertical coordinate y is positive upward (with result $\vec{a} = -g\hat{y}$). We choose time zero to be when the cannon was fired. Then at time zero we have:

$$\vec{a}(0) = -g\hat{y}; \quad \vec{v}(0) = \vec{v}_0; \quad \vec{r}(0) = 0\,. \tag{9}$$

[5]How low is "sufficiently low"? The answer depends upon how precisely you wish to describe the motion, and the relative magnitudes of the force of gravity and the force of the air.

We can get these quantities as a function of time by integrating the acceleration to get the velocity and by integrating the velocity to get the position (see Eq. (8)). The result for the velocity is:

$$\vec{v} = \vec{v}_0 + \int_0^t a(\vec{t}')\, dt' = \vec{v}_0 - \int_0^t g\, dt'\, \hat{y} = \vec{v}_0 - gt\hat{y}\,. \tag{10}$$

In the following three exercises, illustrate by using or changing Fig. 4.

▷ Suppose gravity is turned off. Show that the object would follow a straight-line trajectory at constant speed and that its distance from the origin would increase linearly with time.

▷ suppose that gravity is increased until it is so large that its terms in Eqs. (10) overwhelm the $\vec{v}_0$ terms. Show that under those circumstances the object would appear to simply fall to the ground.

▷ Describe a real projectile's path as being between the trajectories that would result from zero gravity and from very large gravity.

5c. The Example in Cartesian Coordinates. Equations (9) and (10) for ballistic motion can be written in terms of the projectile's non-accelerated x-components and constantly-accelerated y-components:

$$\begin{aligned}
\vec{a}(t) &= -g\hat{y}\,, \\
\vec{v}(t) &= (v_0\, \cos\theta_0)\,\hat{x} + (-gt + v_0 \sin\theta_0)\,\hat{y}\,, \\
\vec{r}(t) &= v_0 t \cos\theta_0 \hat{x} + (-gt^2/2 + v_0 t \sin\theta_0)\,\hat{y}.
\end{aligned}$$

Notice that each component can be integrated separately because the Cartesian unit vectors are independent of time (they stay fixed as time progresses).

5d. Equation of the Path: the Trajectory. The Cartesian trajectory equation, $y(x)$, can be found by eliminating t in the two equations $x(t)$ and $y(t)$. From the above we get:

$$x(t) = v_0 t \cos\theta_0\,,$$

and

$$y(t) = -gt^2/2 + v_0 t \sin\theta_0\,.$$

Solving the first for t and substituting that into the second, gives

$$y = \left[-g/(2v_0^2 \cos^2\theta_0)\right] x^2 + [\tan\theta_0]\, x\,.$$

This is the equation of a parabola, which is indeed the path of a projectile undergoing idealized ballistic motion (see Fig. 4).

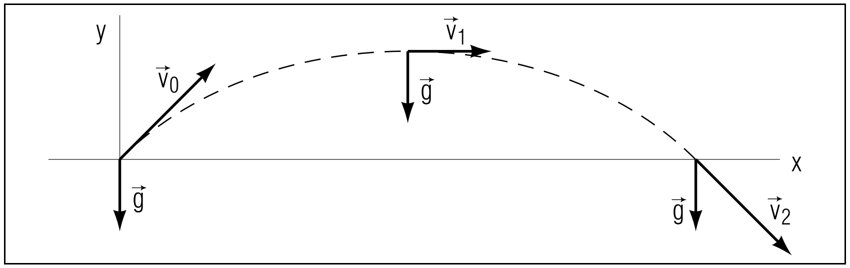


Figure 4. An object in ballistic motion follows a parabolic trajectory. The acceleration and velocity are indicated at three different places. Noticer how the object "falls" as it moves from left to right.

5e. The Range. The trajectory equation, $y(x)$, is useful in answering questions that relate to position coordinates only. Will the object clear a wall? How far will it go? How high will it go? For example, the range R of a ballistic missile is the distance travelled in x before the projectile strikes the ground. In Fig. 4, x is R when $y(x) = 0$. Thus you can set $y(R) = 0$ in the quadratic expression,

$$y(x) = \left[-g/(2v_0^2 \cos^2\theta)\right]\, x^2 + [\tan\theta]\, x\,,$$

and, using the identity $\sin 2\theta = 2\sin\theta\cos\theta$, find:

$$R = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}.$$

► This equation gives $R(v_0, \theta)$. Show that for *any* fixed v_0 the maximum range occurs for $\theta = 45°$.

5f. Maximum Height of a Projectile. The equation of the path can also be used to develop an equation relating the maximum height to the initial velocity of the projectile. At the point of maximum height, $dy/dx = 0$. Differentiating the equation for the trajectory $y(x)$ gives $dy/dx = 0$ at $x = (v_0^2/g)(\sin\theta\cos\theta)$.

► Show that substituting this into the equation for y gives the maximum height as $y_{\max} = (v_0^2 \sin^2\theta)/(2g)$.

Acknowledgments

We wish to thank Kristen Schalm and Prof. James Linnemann for pointing out typos. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

PROBLEM SUPPLEMENT

► If you get really stuck on a problem, turn to the appropriate hint in this module's *Special Assistance Supplement* and then try to continue with the solution. For example, problem 1a, below, has a hint in the box labeled [S-1] in the *Special Assistance Supplement*. If you still can't solve problem 1a after using the hint, the [S-1] box contains a reference to another box containing a further hint.

► Problems 11, 13-15 also occur in this module's *Model Exam*.

► Look out! Units and *variables* are set in different typefaces. *Help: [S-14]*

1. The position of a particle is given by the expression:

$$\vec{r} = (2\,\mathrm{m/s^2})t^2\hat{x} + (3\,\mathrm{m/s})t\hat{y}\,.$$

   a. What is the shape of the path? *Help: [S-1]*

   b. What is the velocity at the point $x = 2\,\mathrm{m}$, $y = 3\,\mathrm{m}$? *Help: [S-2]*

2. Which of the following time-dependent positions are consistent with a constant but non-zero acceleration? *Help: [S-3]*

   a. $(3\,\mathrm{m/s^3})t^3\hat{x} + (1\,\mathrm{m/s^2})t^2\hat{y}$

   b. $(2\,\mathrm{m/s})t\hat{x} + (1\,\mathrm{m})\hat{y}$

   c. $(2\,\mathrm{m/s^2})t^2\hat{x} + (1\,\mathrm{m})\hat{y}$

   d. $(5\,\mathrm{m/s^2})t^2\hat{y}$

   e. $(4\,\mathrm{m})\hat{x} + (3\,\mathrm{m})\hat{y}$

3. Which of the above (a - e, Prob. 2) correspond to constant but non-zero velocity? *Help: [S-4]*

4. Which of the above (a - e, Prob. 2) correspond to an object at rest? *Help: [S-5]*

5. A particle has the following information given about its motion:

x-dir.: A constant velocity of $5\,\mathrm{m/s}$, with initial condition $x_0 = 0\,\mathrm{m}$.

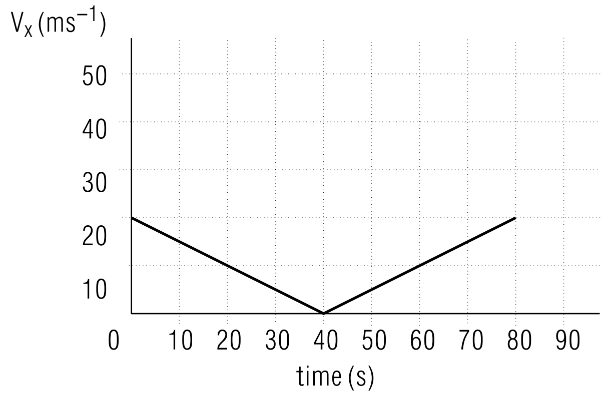
y-dir.: A constant acceleration of $-2\,\mathrm{m/s^2}$, with initial conditions $y_0 = 10\,\mathrm{m}$, $v_{0y} = -3\,\mathrm{m/s}$.

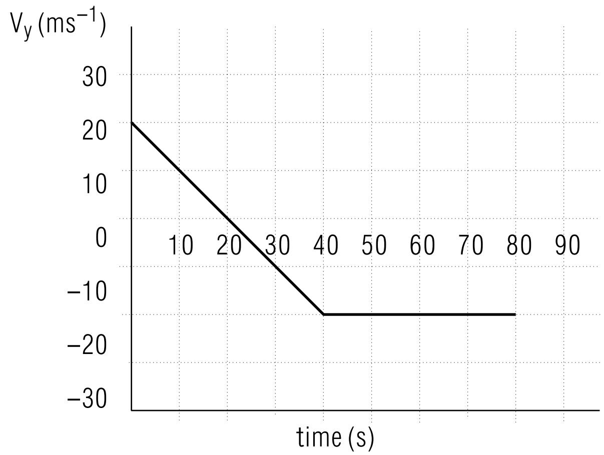
a. Recognizing that these motions fit the special cases of constant velocity and constant acceleration, write the equations $x(t)$, $v_x(t)$, $a_x(t)$, $y(t)$, $v_y(t)$ and $a_y(t)$. *Help: [S-6]*

b. Write the vector equations for $\vec{r}(t)$, $\vec{v}(t)$ and $\vec{a}(t)$. *Help: [S-7]*

6. A particle is at rest at $t = 0$. After $t = 0$, the acceleration of the particle is given by $\vec{a} = (3\,\text{m/s}^2)\hat{x} - (2\,\text{m/s}^2)\hat{y}$. What is the shape of the trajectory? *Help: [S-8]*

7. The x and y components of velocity of a particle are given below.





a. What is the average velocity over the time interval $0 \le t \le 40\,\text{s}$? *Help: [S-9]*

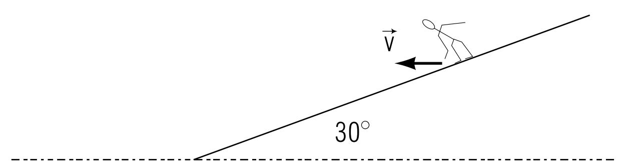
b. What is the acceleration at $t = 20\,\text{s}$? *Help: [S-10]*

c. What is the average acceleration over the interval $20\,\text{s} \le t \le 60\,\text{s}$? *Help: [S-11]*

8. A parachutist jumps from an airplane at a height of 300 m. The parachute immediately opens and she descends at a constant rate of 8 m/s. As she descends, a steady wind of 3 m/s is blowing toward the south. How far south of the point where she left the airplane will the parachutist strike the earth? *Help: [S-12]*

9. A particle starts from rest and moves with constant acceleration $\vec{a} = A\hat{x} + B\hat{y}$, where A and B are constants. Show that the average velocity over the time interval 0 to t is half the instantaneous velocity at t. *Help: [S-13]*

10. A boy stands on an inclined surface which makes an angle of 30° with the horizontal (see sketch). He throws a ball so that it leaves his hand horizontally, down near the surface of the incline, with a speed of 10 m/s. At what distance down the incline (measured from the boy) will the ball strike the incline? (Assume he releases the ball from a point at the surface and assume that the only acceleration is that of gravity.)



11. The velocity of a particle is given by the expression:

$$\vec{v} = [2(t/\,\text{s})\hat{x} + \hat{y}]\,\text{m/s}\,.$$

a. At $t = 0$ the particle is at $x = 0$, $y = 0$. What is the position of the particle at a later time t?

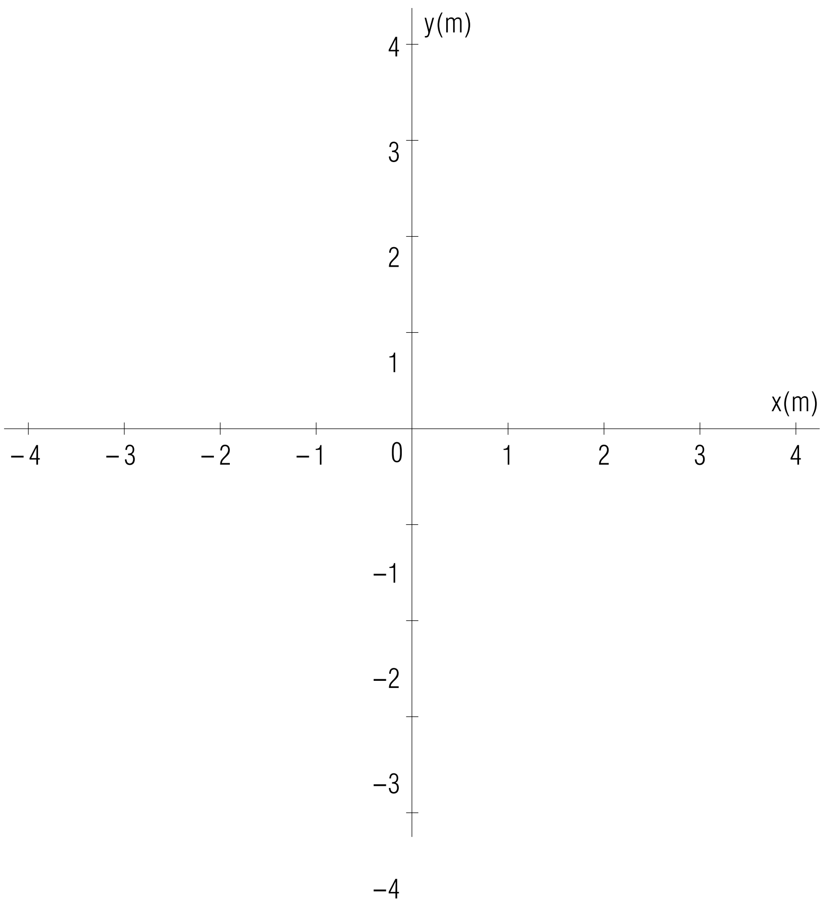
b. What is the average velocity over the interval $t = 2\,\text{s}$ to $t = 3\,\text{s}$?

12. A particle moves with position vector

$$\vec{r} = \left[\left(\frac{t}{\text{s}}\right)\hat{x} + \left(\frac{4}{3}\frac{t^3}{\text{s}^3} - 4\frac{t}{\text{s}}\right)\hat{y}\right]\,\text{m}\,.$$

a. Plot the path on the graph below. Hint: First, fill in the table. *Help: [S-15]*

t(s)	x(m)	y(m)
−2.1	−2.1	−3.9
−2.0		
−1.0		
0.0		
1.0		
2.0		
2.1		

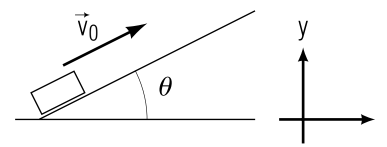


b. On the same curve, plot the velocity at $t = 1\,\mathrm{s}$. The scale for plotting the velocity can be any you choose. A convenient scale for velocity is 1 unit of velocity = 1/2 unit of length. *Help: [S-16]*

c. On the same curve, plot the velocity at $t = -2\,\mathrm{s}$. *Help: [S-17]*

d. On the same curve, plot the acceleration at $t = -1\,\mathrm{s}$. *Help: [S-18]*

13. A block is projected up an inclined surface which makes an angle θ with the horizontal. The initial speed is v_0 and the inclined surface is frictionless. Use a coordinate system defined by x positive to the right in a horizontal direction, y positive up (as illustrated).



a. What is the initial velocity?

b. The acceleration of the block is observed to be:

$$\vec{a} = -g \sin\theta\, \cos\theta\, \hat{x} - g \sin^2\theta\, \hat{y}.$$

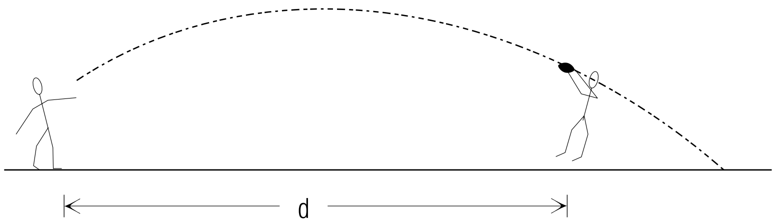
Develop expressions for $x(t)$ and $y(t)$. Divide one by the other to check that they (properly) predict that y(x) is a straight-line function with slope $\tan\theta$.

14. A particle moves with acceleration:

$$\vec{a} = \left[-6\,(t/\,\mathrm{s})^2\hat{x} + (t/\,\mathrm{s})\hat{y}\right]\ \mathrm{m/s^2}\,.$$

At $t = 0$ the particle is at rest at the origin. What is the equation of the path (i.e., the trajectory)?

15. A football is thrown by a quarterback at a speed of $20\,\mathrm{m/s}$ at an angle of $45°$ with the horizontal. A receiver is running such that he will pass under the ball downfield from the point where it was released. Assuming the quarterback and the receiver are the same height, and the receiver can jump or reach one meter higher than the point of release by the quarterback, what is the minimum distance from the quarterback the ball can be caught?



Brief Answers:

If you do not understand an answer, refer to the last comment (after the "Hints") in the Special Assistance Supplement for that problem.

1. a. $x = (2\,\mathrm{m/s^2})t^2$; $y = (3\,\mathrm{m/s})t$;
   $t = y/(3\,\mathrm{m/s})$;
   $x = (2\,\mathrm{m/s^2}) \cdot [y/(3\,\mathrm{m/s})]^2 = [2/(9\,\mathrm{m})]\,y^2$, which is a parabola.

   b. From $x = (2\,\mathrm{m/s^2})t^2$, $y = (3\,\mathrm{m/s})t$, the particle will pass through the point $x = 2\,\mathrm{m}$, $y = 3\,\mathrm{m}$ at $t = 1\,\mathrm{s}$. From part (a), $\vec{v}(1\,\mathrm{s}) = (4\hat{x} + 3\hat{y})\,\mathrm{m/s}$.

2. In order to have a constant acceleration, $d^2\vec{r}/dt^2 =$ constant.

   a. $\dfrac{d\vec{r}}{dt} = \vec{v} = [9\dfrac{t^2}{\mathrm{s}^2}\hat{x} + 2\dfrac{t}{\mathrm{s}}\hat{y}]\,\mathrm{m/s}$
   $\dfrac{d^2\vec{r}}{dt^2} = \dfrac{d\vec{v}}{dt} = \vec{a} = [18\dfrac{t}{\mathrm{s}}\hat{x} + 2\hat{y}]\,\mathrm{m/s^2}$: This is not constant in time.

   b. $\vec{v} = 2\hat{x}\,\mathrm{m/s}$; $\vec{a} = 0$. This is constant, but zero.

   c. $\vec{v} = 4t\hat{x}\,\mathrm{m/s^2}$; $\vec{a} = 4\hat{x}\,\mathrm{m/s^2}$. This is constant, and $\neq$ zero.

   d. $\vec{v} = 10t\hat{y}\,\mathrm{m/s^2}$; $\vec{a} = 10\hat{y}\,\mathrm{m/s^2}$. This is constant and $\neq$ zero.

   e. $\vec{v} = 0$; $\vec{a} = 0$. This is a zero acceleration and a zero velocity.

3. Answer: b.

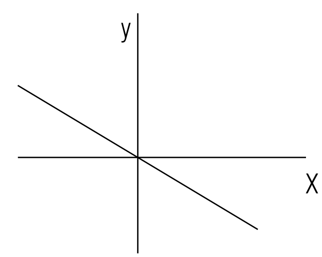
4. Answer: e.

5. a. $a_x = 0$; $v_x = 5\,\mathrm{m/s}$; $x = 5\,\mathrm{m/s}\,t$;

   $a_y = -2\,\mathrm{m/s^2}$; $v_y = -3\,\mathrm{m/s} - (2\,\mathrm{m/s^2})t$;
   $y = 10\,\mathrm{m} - (3\,\mathrm{m/s})t - (2\,\mathrm{m/s^2})t^2/2$

   b. $\vec{r}(t) = \hat{x}x(t) + \hat{y}y(t)$ with $x(t)$ and $y(t)$ given in part a. $\vec{v}(t) = \hat{x}v_x(t) + \hat{y}v_y(t)$, etc.

6. Given: $\vec{v}_0 = 0$. Choose the origin so $\vec{r}_0 = 0$. Then:
   $\vec{v} = (3t\hat{x} - 2t\hat{y})\,\mathrm{m/s^2}$;
   $\vec{r} = \left[(3/2)t^2\hat{x} - t^2\hat{y}\right]\,\mathrm{m/s^2}$.
   From this,
   $x = (3/2)t^2\,\mathrm{m/s^2}$;
   $y = -t^2\,\mathrm{m/s^2}$;
   $x = -(3/2)y$.
   Straight line slope $= -2/3$.

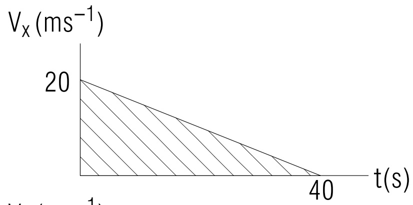


7. a. $\vec{v}_{av} = (\Delta x/\Delta t)\hat{x} + (\Delta y/\Delta t)\hat{y}$.

   The quantities Δx and Δy are the respective areas under the $v(t)$ curves for each component. The area is just the area of a triangle.

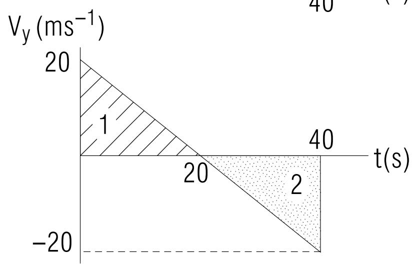
   Δx:

   $$\begin{aligned}\text{Area} &= \Delta x\\ &= (40\,\text{s})(20\,\text{m/s})/2\\ &= 400\,\text{m}.\end{aligned}$$



   Δy:

   $$\begin{aligned}\text{Area 1} &= \text{displacement}\\ &\quad (0 \text{ to } 20\,\text{s})\\ &= (20\,\text{s})(20\,\text{m/s})/2\\ &= 200\,\text{m}.\\ \text{Area 2} &= \text{displacement}\\ &\quad (20\,\text{s to } 40\,\text{s})\\ &= (20\,\text{s})(-20\,\text{m/s})/2\\ &= -200\,\text{m}.\end{aligned}$$



   Total Area = Area 1 + Area 2 = 0.

   $\vec{v}_{av} = (400\,\text{m}/40\,\text{s})\hat{x} + (0\,\text{m}/40\,\text{s})\hat{y} = 10\hat{x}\,\text{m/s}$.

   b. $\vec{a}(20\,\text{s}) = (dv_x/dt)\hat{x} + (dv_y/dt)\hat{y}$.
   Using the fact that the derivative of a straight line is the slope of the line, $\vec{a}(20\,\text{s}) = [-(1/2)\hat{x} - \hat{y}]\ \text{m/s}^2$

   c. $\vec{a}_{av} = \Delta v/\Delta t = [\vec{v}(60\,\text{s}) - \vec{v}(20\,\text{s})]\ /\ [60\,\text{s} - 20\,\text{s}]$
   $\vec{v}$ at $t = 60 units$ is (from the graph) $(10\hat{x} - 20\hat{y})\,\text{m/s}$.
   $\vec{v}$ at $t = 20\,\text{s}$ is $(10\hat{x} + 0\hat{y})\,\text{m/s}$.
   $\vec{a}_{av} = -(1/2)\hat{y}\,\text{m/s}^2$.

8. Let y be positive down, x be positive south.

$a_x = 0$	$a_y = 0$
$v_x = 3\,\text{m/s}$	$v_y = 8\,\text{m/s}$
$x = 3\,\text{m/s}t$	$y = 8\,\text{m/s}t$

   The time of descent is given by putting $y = 300\,\text{m}$, and solving for t: $t = 37.5\,\text{s}$. The displacement is obtained by putting $t = 37.5\,\text{s}$ into the equation for x: $x = 112.5\,\text{m}\,\text{south}$.

9. $$\begin{aligned}\vec{v}(t) &= \int_0^t \vec{a}(t')\,dt' + \vec{v}_0\\ &= \hat{x}\int_0^t A\,dt' + \hat{y}\int_0^t B\,dt' + 0\end{aligned}$$

$$\begin{aligned} &= At\hat{x} + Bt\hat{y} = (A\hat{x} + B\hat{y})t\\ \vec{v}_{av} &= \frac{\int_0^t \vec{v}(t')\,dt'}{t-0}\\ &= \frac{\hat{x}}{t}\left(\frac{1}{2}At^2\right) + \frac{\hat{y}}{t}\left(\frac{1}{2}Bt^2\right)\\ &= \frac{1}{2}(A\hat{x} + B\hat{y})t = \frac{1}{2}\vec{v}(t)\end{aligned}$$

This is a very special result that holds true only for constant acceleration from rest. Any other set of conditions must be treated by the above method to find any special relation between $\vec{v}_{av}$ and $\vec{v}(t)$.

10. Choose a coordinate system with axes parallel and perpendicular to the inclined surface. Choose this coordinate system because then it is easy to state the mathematical condition corresponding to the ball hitting the incline: it is when (again) $y = 0$. In such a system the acceleration of the ball, the acceleration of gravity, is: $\vec{a} = \vec{g} = -g\sin 30^\circ\hat{x} - g\cos 30^\circ\hat{y}$. *Help: [S-41]*
    Using the equations for constant acceleration (the origin is at the boy):

$$\begin{aligned} a_x &= -g\sin 30^\circ\\ v_x &= -g\sin 30^\circ t - v_0\cos 30^\circ\\ x &= -(g/2)\sin 30^\circ t^2 - v_0\cos 30^\circ t\\ a_y &= -g\cos 30^\circ\\ v_y &= -g\cos 30^\circ t + v_0\sin 30^\circ\\ y &= -(g/2)\cos 30^\circ t^2 + v_0\sin 30^\circ t\end{aligned}$$

    Putting $y = 0$ and solving for t gives two solutions: either $t = 0$ or $t = 1.18\,\text{s}$. There are two times when the ball is at $y = 0$: when it is released and when it strikes the surface. The time in question is $t = 1.18\,\text{s}$. At this time, $x = -13.62\,\text{m}$.

11. a. $\vec{r}(t) = [(t/\,\text{s})^2\hat{x} + (t/\,\text{s})\hat{y}]\,\text{m}$

    b. $\vec{v}_{av} = (5\hat{x} + \hat{y})\,\text{m/s}$

12. a. $x = t\,\text{m/s}$

$$y = \left[\left(\frac{4}{3}\right)\frac{t^3}{\text{s}^2} - 4t\right]\ \text{m/s}\ .$$

Completing the table:

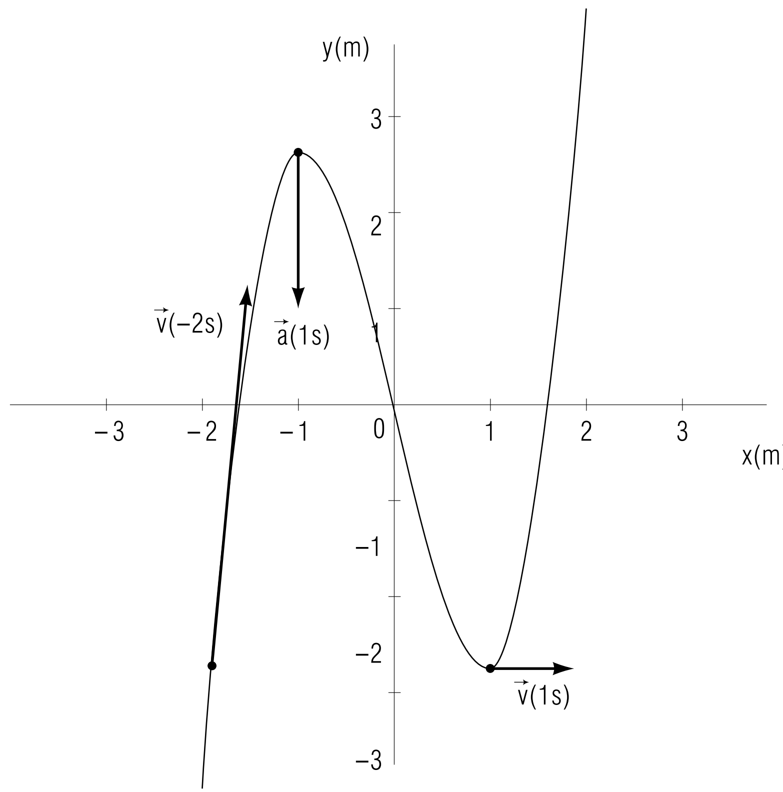
t(s)	x(m)	y(m)
−2.1	−2.1	−3.9
−2.0	−2.0	−2.7
−1.0	−1.0	2.7
0.0	0.0	0.0
1.0	1.0	−2.7
2.0	2.0	2.7
2.1	2.1	3.9

See the graph in part (d).

b. $\vec{v} = \left[\hat{x} + \left(4\frac{t^2}{\mathrm{s}^2} - 4\right)\hat{y}\right]$ m/s. $\vec{v}(1\,\mathrm{s}) = \hat{x}\,\mathrm{m/s}$. At $t = 1\,\mathrm{s}$, the particle is at $x = 1\,\mathrm{m}$, $y = -2.7\,\mathrm{m}$. The velocity at this time is indicated on the graph. The point $x = 1\,\mathrm{m}$, $y = -2.7\,\mathrm{m}$ is the turn around point and the slope $dy/dx = 0$. The velocity is tangent to the curve.

c. $\vec{v}(-2\,\mathrm{s}) = (\hat{x} + 12\hat{y})\,\mathrm{m/s}$. The particle is at $x = -2\,\mathrm{m}$, $y = -2.7\,\mathrm{m}$ and the velocity is as indicated. The velocity is tangent to the curve.

d. $\vec{a} = 8t\hat{y}\,\mathrm{m/s^3}$. At $t = -1\,\mathrm{s}$, $\vec{a} = -8\hat{y}\,\mathrm{m/s^2}$, $\vec{v} = \hat{x}\,\mathrm{m/s}$ and $\vec{r} = (-\hat{x} + 2.7\hat{y})\,\mathrm{m}$. The velocity is horizontal at this point, and the acceleration is vertical. Have you noted that v_x = constant in this problem? See sketch below.



13. a. $\vec{v}_0 = v_0\cos\theta\hat{x} + v_0\sin\theta\hat{y}$

b. $v = -gt\sin\theta\cos\theta\hat{x} - gt\sin^2\theta\hat{y} + v_0\cos\theta\hat{x} + v_0\sin\theta\hat{y}$

With the origin chosen so that $\vec{r}_0 = 0$:

$x = -\frac{1}{2}gt^2\sin\theta\,\cos\theta + v_0t\cos\theta,$

$y = -\frac{1}{2}gt^2\sin^2\theta + v_0t\sin\theta,$

$x/\cos\theta = -\frac{1}{2}gt^2\sin\theta + v_0t$, and

$y/\sin\theta = -\frac{1}{2}gt^2\sin\theta + v_0t.$

Therefore, $\frac{x}{\cos\theta} = \frac{y}{\sin\theta}$, $y = x\tan\theta$, a straight line.

14. $\vec{r}_0 = 0$; $\vec{v}_0 = 0$; $v = [-2(t/\text{s})^3\hat{x} + \frac{(t/\text{s})^2}{2}\hat{y}]\,\text{m/s}$;

$x = -\frac{(t/\text{s})^4}{2}\,\text{m}$;

$y = \frac{(t/\text{s})^3}{6}\,\text{m}$;

$x = -\frac{(6y/\text{m})^{4/3}}{2}\,\text{m}$

or: $(x/\text{m})^3 = -162(y/\text{m})^4$

15. Choose the origin at the point of release, choose $\hat{y}$ upward.

$a_x = 0$; $v_x = v_{0,x} = 20\,(\text{m/s})\cos 45°$;

$x = 20(\cos 45°)\,\text{m}(t/\text{s})$; $a_y = -g$; $v_y = v_{0,y} - gt$

$= [20\sin 45° - 9.8(t/\text{s})]\,\text{m/s}$;

$y = \left[-4.9(t/\text{s})^2 + 20\sin 45°(t/\text{s})\right]\,\text{m}$.

The range for the ball to be caught is $y \leq 1\,\text{m}$.

Putting $y = 1\,\text{m}$ into the equation gives these two solutions: $t = 2.81\,\text{s}$ and $t = 0.072\,\text{s}$. *Help: [S-42]*

At $t = 2.81\,\text{s}$, $x = 39.7\,\text{m}$, but at $t = 0.072\,\text{s}$, $x = 1.02\,\text{m}$. Picture each of these in your mind's eye!

SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from PS-1a)*

Find $x(t)$, $y(t)$ for the given $\vec{r}(t)$. *Help: [S-19]*

S-2 *(from PS-1b)*

Find the time when $x = 2\,\text{m}$. This is the same time as when $y = 3\,\text{m}$.

S-3 *(from PS-2)*

$\vec{a} = d^2\vec{r}/dt^2$. *Help: [S-20]*

S-4 *(from PS-3)*

$\vec{v} = d\vec{r}/dt$.

S-5 *(from PS-4)*

The object is at rest if the coordinates describing its position do not change in time. *Help: [S-21]*

S-6 *(from PS-5a)*

Partial information about each component is given. Solve for the motion of each component vector separately. *Help: [S-22]*

S-7 *(from PS-5b)*

Using vector rotation, add the components of part (a).

S-8 *(from PS-6)*

The trajectory is determined by the curve of $y(x)$ or (equivalently) $x(y)$. *Help: [S-23]*

S-9 *(from PS-7a)*

Average velocity is defined as: $\int_{t_1}^{t_2} \vec{v}(t')\,dt'/(t_2 - t_1) = \Delta\vec{r}/\Delta t$. *Help: [S-24]*

S-10 *(from PS-7b)*

See problem 1(b). *Help: [S-25]*

S-11 *(from PS-7c)*

See problem 1(b). *Help: [S-26]*

S-12 *(from PS-8)*

$a_x = 0$ and $a_y = 0$. *Help: [S-27]*

S-13 *(from PS-9)*

This is a constant acceleration with x- and y-components. *Help: [S-28]*

S-14 *(from PS-head)*

For example, in $(t/\,\mathrm{s})$ the "t" is the variable "time" and the "s" is the unit "seconds." For example: if $t = 3\,\mathrm{s}$, then $(t/\,\mathrm{s}) = (3\,\mathrm{s})/(\,\mathrm{s}) = 3$.

S-15 *(from PS-11a)*

The path has zero slope at $t = \pm 1\,\mathrm{s}$.

S-16 *(from PS-11b)*

$\vec{v} = \left[\hat{x} + \left(4\dfrac{t^2}{\mathrm{s}^2} - 4\right)\hat{y}\right]$ m/s. *Help: [S-29]*

S-17 *(from PS-11c)*

See method of part (b).

S-18 *(from PS-11d)*

$\vec{a} = 8t\hat{y}\,\mathrm{m/s^3}$.

S-19 *(from [S-1])*

Solve for $t(y)$ and substitute into $x(t)$.

S-20 *(from [S-3])*

$\vec{a} = d\vec{v}/dt = (d/dt)(d\vec{r}/dt)$. If these give trouble, review how to differentiate a vector.

S-21 *(from [S-5])*

To meet the above requirement, $\vec{v} = 0$ and $\vec{a} = 0$ for all times.

S-22 *(from [S-6])*

The motion of each vector is one dimensional. *Help: [S-30]*

S-23 *(from [S-8])*

x and y are components of $\vec{r}$. *Help: [S-31]*

S-24 *(from [S-9])*

$\int_{t_1}^{t_2} v_x\,dt'$ = area under the $v_x(t)$ vs. t curve from t_1 to t_2. *Help: [S-32]*

S-25 *(from [S-10])*

$\vec{a} = d\vec{v}/dt$. *Help: [S-33]*

S-26 *(from [S-11])*

$\vec{a}_{av} = [\vec{v}(60\,s) - \vec{v}(20\,\mathrm{s})]\,/(60\,\mathrm{s} - 20\,\mathrm{s})$,
$\vec{a}_{av} \neq [\vec{a}(60\,\mathrm{s}) + \vec{a}(20\,\mathrm{s})]\,/2$. *Help: [S-34]*

S-27 *(from [S-12])*

Both v_x and v_y are constant. *Help: [S-35]*

S-28 *(from [S-13])*

$v_x = At$. *Help: [S-36]*

S-29 *(from [S-16])*

$\vec{v}(1\,\mathrm{s}) = \hat{x}\,\mathrm{m/s}$.

S-30 *(from [S-22])*

Some equations for motion in one dimension with constant acceleration are: $x = x_0 + v_0 t + at^2/2$, $v = v_0 + at$, $v^2 = v_0^2 + 2a(x - x_0)$.

S-31 *(from [S-23])*

$\vec{r}(t) = \int_0^t dt' \int_0^{t'} \vec{a}(t'')\,dt'' + \vec{r}_0$. *Help: [S-37]*

S-32 *(from [S-24])*

The area of a triangle is $(1/2)bh$. *Help: [S-38]*

S-33 *(from [S-25])*

At $t = 20\,\mathrm{s}$, $dv_x/dt = -(20/40)\,\mathrm{m/s^2} = -(1/2)\,\mathrm{m/s^2}$. This problem requires a derivative to be taken from a curve, which is the slope of the line at the time given.

S-34 *(from [S-26])*

$\vec{v}(60\,\mathrm{s}) = (10\hat{x} - 20\hat{y})\,\mathrm{m/s}$. In this problem, $\vec{v}(t)$ must be found from the graphical representations of the components. If you had trouble with parts (b) and (c), review problem 1(b).

S-35 *(from [S-27])*

The parachutist will reach the ground in 37.5 s.

S-36 *(from [S-28])*

$v_{x,av} = \dfrac{1}{t}\int_0^t A\,t'\,dt'$.

S-37 *(from [S-31])*

For the particle to be at rest at $t = 0$, $\vec{v} = 0$. *Help: [S-39]*

S-38 *(from [S-32])*

If $v_y < 0$, area has a negative sign, indicating a displacement in the negative direction.

S-39 *(from [S-37])*

For simplicity, put $\vec{r}_0 = 0$. *Help: [S-40]*

S-40 *(from [S-39])*

$\vec{v} = (3\,\mathrm{m/s^2})t\hat{x} - (2\,\mathrm{m/s^2})t\hat{y}$ and $\vec{r}(t) = \int_0^t v(t')\,dt'$. If this problem gives trouble, work on how to make transformations of the type $x(t) \leftrightarrow t(x)$.

S-41 *(from PS-10)*

Draw a graph with a single vector extending out from the origin at an angle of 240° CCW from the positive x-axis. Label this vector $\vec{g}$. Then: $\vec{g} = g\cos 240^\circ\,\hat{x} + g\sin 240^\circ\,\hat{y}$. Alternatively, simply look at the diagram and write: $\vec{g} = -g\sin 30^\circ\,\hat{x} - g\cos 30^\circ\,\hat{y}$.

S-42 *(from PS-15)*

We let $b \equiv t/\,\mathrm{s}$ so the y-equation above becomes:

$$4.9b^2 - 14.4b + 1.0 = 0\,.$$

Then solving this quadratic equation (see any high school or college algebra book):

$$b = \frac{14.4 \pm \sqrt{(-14.4)^2 - 4(4.9)(1.0)}}{(2)(4.9)}\,.$$

MODEL EXAM

1. See Output Skills K1-K2 on this module's *ID Sheet*.

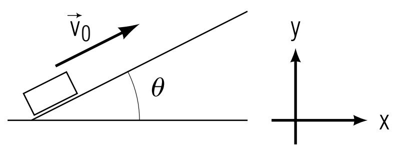
2. The velocity of a particle is given by the expression:

$$\vec{v} = [2(t/\,\mathrm{s})\hat{x} + \hat{y}]\,\mathrm{m/s}.$$

   a. At $t = 0$ the particle is at $x = 0$, $y = 0$. What is the position of the particle at a later time t?

   b. What is the average velocity over the interval $t = 2\,\mathrm{s}$ to $t = 3\,\mathrm{s}$?

3. A block is projected up an inclined surface which makes an angle θ with the horizontal. The initial speed is v_0 and the inclined surface is frictionless. Use a coordinate system defined by x positive to the right in a horizontal direction, y positive up (as illustrated).



   a. What is the initial velocity?

   b. The acceleration of the block is observed to be:

$$\vec{a} = -(g\sin\theta\,\cos\theta)\,\hat{x} - (g\sin^2\theta)\,\hat{y}.$$

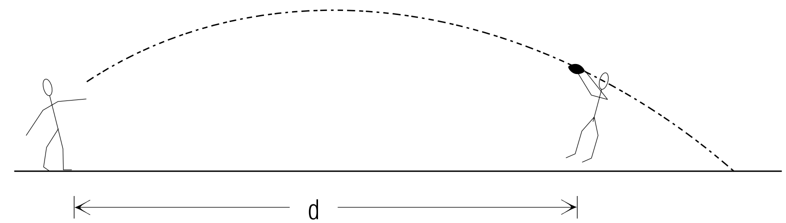
   Develop expressions for $x(t)$ and $y(t)$. Divide one by the other to check that they (properly) predict that y(x) is a straight-line function with slope $\tan\theta$.

4. A particle moves with acceleration:

$$\vec{a} = [-6\,(t/\,\mathrm{s})^2\hat{x} + (t/\,\mathrm{s})\hat{y}]\,\mathrm{m/s^2}.$$

   At $t = 0$ the particle is at rest at the origin. What is the equation of the path (i.e., the trajectory)?

5. A football is thrown by a quarterback at a speed of $20\,\mathrm{m/s}$ at an angle of $45°$ with the horizontal. A receiver is running such that he will pass under the ball downfield from the point where it was released. Assuming the quarterback and the receiver are the same height, and the receiver can jump or reach one meter higher than the point of release by the quarterback, what is the minimum distance from the quarterback the ball can be caught?



Brief Answers:

1. See this module's *text*.
2. See this module's *Problem Supplement*, problem 11.
3. See this module's *Problem Supplement*, problem 13.
4. See this module's *Problem Supplement*, problem 14.
5. See this module's *Problem Supplement*, problem 15.