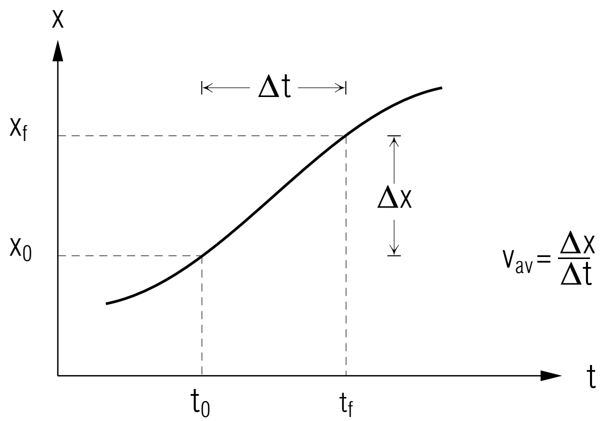
PHYSNET

MISN-0-7

KINEMATICS IN ONE DIMENSION



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

KINEMATICS IN ONE DIMENSION

by

Leon F. Graves

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Input Skills:

1. Express physical quantities in the proper units with the appropriate number of significant digits (MISN-0-403).
2. Manipulate units of physical measurement by means of algebra (MISN-0-403).
3. Differentiate and integrate polynomials, sines and cosines (MISN-0-1).
4. Express a vector in terms of its magnitude and direction (MISN-0-2).
5. Draw and measure simple graphs (MISN-0-401).

Output Skills (Knowledge):

K1. Vocabulary: average velocity, instantaneous velocity, speed, average acceleration, instantaneous acceleration.

Output Skills (Problem Solving):

S1. Given a particle's position function as a table, graph or mathematical function of time, determine its average velocity during a specified time interval and its instantaneous velocity at a specified time. Estimate its acceleration during a specified time interval and its instantaneous acceleration at a specified time.

S2. Given a particle's acceleration function and its velocity and position at specified times, determine its velocity and position at other times.

Post-Options:

1. "Kinematics of Motion in Two Dimensions" (MISN-0-8).

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New authors, reviewers and field testers are welcome.

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1. Introduction

1a. Kinematics. Kinematics is the study of the motion of particles in terms of space and time. By a particle we mean an identifiable physical object with spatial dimensions so small so that it can be located at a point in a coordinate system.

1b. The Reason for One Dimension. The real world consists of three space-dimensions but in this module we will be dealing only with those motions that are one-dimensional, motions that are along a straight line. This is because motion in a straight line is the simplest motion to analyze so its study is a good introduction to motion in general. Furthermore, when motion does occur in more than one dimension, one often solves for the Cartesian components of the vector quantities. The equations for these Cartesian components have much in common with their one-dimensional counterparts that you will see in this module.

A major reason that it is easier to begin with one-dimensional motion is that one does not have to have a multitude of vector symbols obscuring the other concepts that are being introduced. To get rid of vectors, we always choose a coordinate system in which the straight-line motion being examined is along a coordinate axis. Then there is only one common unit vector and it multiplies all terms in all vector equations, so it can be

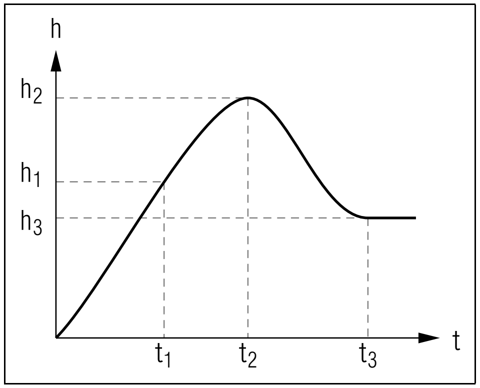


Figure 1. Height of the bottom of a flag, as a function of time, as it is being raised and then lowered to half mast.

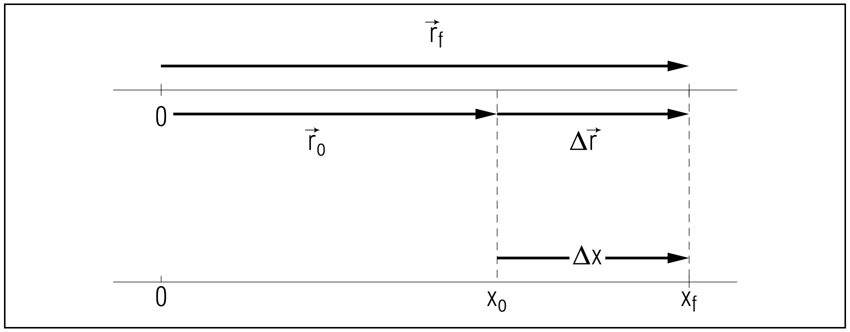


Figure 2. Illustration of displacement quantities (see text).

eliminated ("canceled") from the equations. Although we will thus not use vectors much in one dimension, we suggest that when interpreting positive and negative values for quantities that have direction, you think of those values as being multiplied by the appropriate unit vectors.

2. Position, Displacement

2a. Introduction. In straight line motion, position is defined as distance along the line of motion as measured from some chosen origin. For example, when a flag is run up a flagpole, the position of the bottom of the flag can be taken as its distance above the ground. This position can be shown by a graph of height versus time (see Fig. 1). In this diagram the bottom of the flag reaches height h_1 at time t_1 and h_2 at t_2; it is then lowered, reaching h_3 at t_3, after which it remains at the half-mast position. Since the selection of the coordinate system and its origin is arbitrary, position may be negative or positive in value. The standard SI unit of length is the meter, where 1 meter equals 3.28 feet or 1.09 yards.

2b. Displacement is Change of Position. Position is a vector quantity; for example, $\vec{r} = x\hat{x}$. Displacement, written $\Delta\vec{r}$, is defined as change in position. For example,

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_o = (x_f - x_o)\,\hat{x} = \hat{x}\Delta x\,, \tag{1}$$

where the subscript f indicates final position and the subscript o indicates starting or originating position for the time interval $t_f - t_o$, and $\hat{x}$ is a unit vector in the positive x-direction (see Fig. 2).

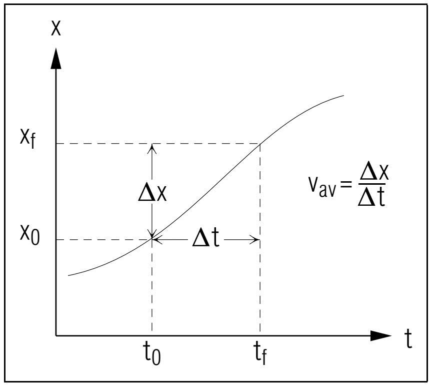


Figure 3. Illustration of quantities used to find average velocity (see text).

3. Velocity

3a. Overview. Velocity is the time rate of change of position. When we change position, we move. We may move slowly or rapidly. We may move forward or backward. Mathematically, velocity is the rate at which one's position changes. Since the rate at which position changes can itself be continually changing, velocity can be different at each instant of time (think of a car speedometer that is continually changing). When beginning to study physics, it is sometimes quite difficult to imagine a quantity as being defined for an infinite continuum of instants during a finite interval of time. In fact, Newton invented calculus just so he could deal with the real world's infinite continuum of instants. To make things a little easier, we will first deal with a finite number of average quantities, then graduate to the real thing.

3b. Average Velocity. If a particle is at position $(x_o\hat{x})$ at time t_o and at position $(x_f\hat{x})$ at a later time t_f, the average velocity over the time interval is (see Fig. 3):

$$\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t} = \frac{(x_f - x_o)\,\hat{x}}{t_f - t_o} = \hat{x}\frac{\Delta x}{\Delta t}\,. \tag{2}$$

3c. Instantaneous Velocity and Speed from $x(t)$. The instantaneous velocity, called simply "the velocity," is the limit of the average velocity as the length of the time interval over which one is averaging approaches zero; that is, as t_f approaches t_o. Dropping the unit vectors in Eq. (2) and taking the limit, we get:

$$v = \lim_{\Delta t \to 0} v_{av} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \equiv \frac{dx}{dt}\,. \tag{3}$$

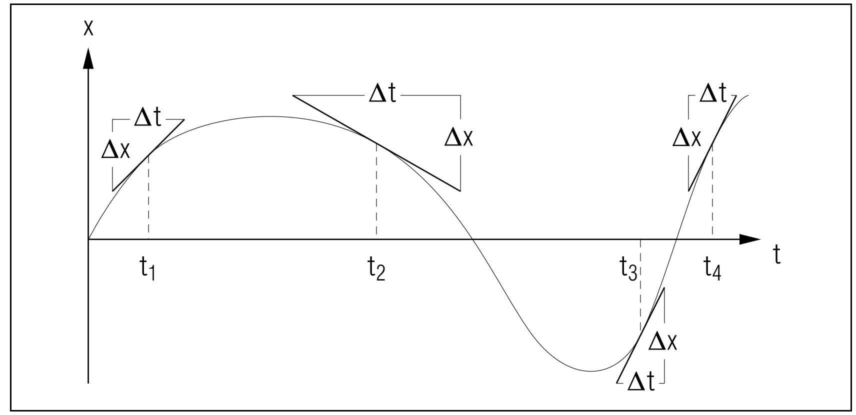


Figure 4. The small triangles show how to measure Δx and Δt to determine instantaneous velocities $\Delta x/\Delta t$ at times t_1, t_2, t_3, t_4.

The always-positive magnitude of v, written $|v|$, is the instantaneous speed, or simply "the speed." It is the quantity a car's speedometer is designed to display, in miles per hour and/or kilometers per hour. The international standard (SI) unit of speed is meters per second.

3d. Instantaneous Velocity From Position Graph. If a graph of position versus time is constructed from a data table, or drawn by a recording instrument, the velocity at any time can be found graphically. The slope of the tangent to the curve at any particular point is dx/dt at that point and this is the instantaneous velocity at that time.

This tangent to the curve can be called the physical slope to distinguish it from a geometrical slope measured in degrees or radians. Unlike a geometrical slope, a physical slope has units determined by the scale of the graph, those of the ordinate divided by those of the abscissa. These slopes can be determined by drawing tangents to the curve at points on the curve, and subsequently using the tangents as the hypotenuses of right triangles that can be drawn and measured (see Fig. 4).

3e. Units. The standard SI unit for speed and velocity is one meter per second, which is approximately equal to 3.28 feet/second or 2.24 mph—a brisk walking speed. To run a four minute mile, a track star must average 22 ft/s or 15 mph (the maximum speed posted for many school zones). In SI units this is 6.70 m/s. Tropical storms are called hurricanes as soon as their winds reach 33 SI units, 33 m/s, equivalent to 64 knots or 74 mph.

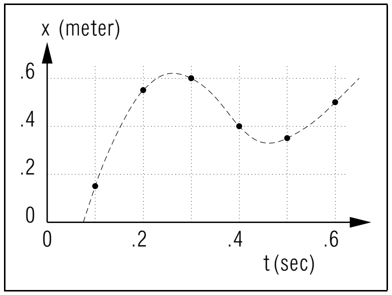


Figure 5. Table graph.

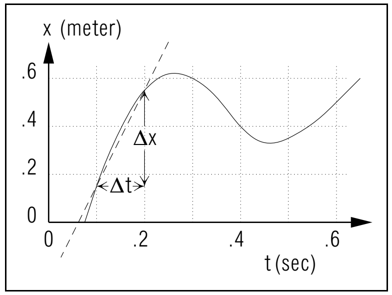


Figure 6. Getting $v_{\rm av}$.

The speed of sound is approximately 330 SI units, 330 m/s.

3f. Example. The motion of a particle traveling along a straight line can be described roughly by giving its position at a number of times. Here is an example:

$t(\mathrm{s})$	0.10	0.20	0.30	0.40	0.50	0.60
$x(\mathrm{m})$	0.15	0.55	0.60	0.40	0.35	0.50

This information can also be shown by plotting a graph, as in Fig. 5. Since we believe such a particle travels smoothly, we would normally connect the points by a smooth line as indicated. In any case, if we collected more and more data on the particle, we could plot more and more points until the graph took on a smooth appearance as in Fig. 6.

Now suppose we need to find the average velocity over the interval from $t = 0.10\,\mathrm{s}$ to $0.20\,\mathrm{s}$. We can use data table to find:

$$\begin{aligned} v_{av} &= \frac{\Delta x}{\Delta t} = \frac{0.55\,\mathrm{m} - 0.15\,\mathrm{m}}{0.20\,\mathrm{s} - 0.10\,\mathrm{s}} \\ &= 4.0\,\mathrm{m/s}\,. \end{aligned}$$

Or we can measure on our (carefully constructed) graph (Fig. 6) to discover that:

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{0.40\,\mathrm{m}}{0.10\,\mathrm{s}} = 4.0\,\mathrm{m/s}\,.$$

This is the slope of the dashed line connecting the end points of the interval in Fig. 6.

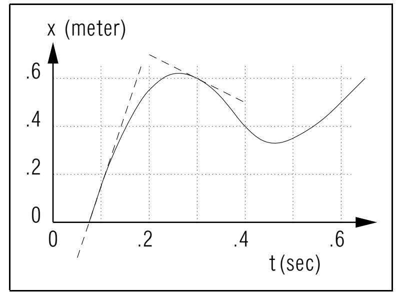


Figure 7. Getting $v(t)$.

On the other hand, if we want the instantaneous velocity at $t = 0.10\,\mathrm{s}$, we let the Δt in Fig. 6 shrink toward zero:

$$v(0.10\,\mathrm{s}) = \lim_{\Delta t \to 0} \left.\frac{\Delta x}{\Delta t}\right|_{0.10\,\mathrm{s}} = \left.\frac{\Delta x}{\Delta t}\right|_{0.10\,\mathrm{s}}\,.$$

which is just the slope of the first dashed line in Fig. 7. That is, the (instantaneous) velocity at any given time is the slope of the graph, the time derivative of the function, at that time.

We can immediately see from Fig. 7 that v is positive throughout the interval from $t = 0.10\,\mathrm{s}$ to $0.20\,\mathrm{s}$ (for example), because x is always increasing with t throughout this interval.

4. Acceleration

4a. Overview. The word "acceleration" implies a change in velocity. Thus we must associate acceleration with change in velocity over some interval of time; we must not associate it with any one particular instantaneous velocity. Both direction and magnitude of velocity change are important. For example, a ball thrown upward into the air slows down, momentarily stops, then picks up downward velocity, all because of the constant downward acceleration due to gravity.

4b. Average Acceleration. If a particle has a velocity $v_0\hat{x}$ at time t_0, and a velocity $v_f\hat{x}$ at a later time t_f, the average acceleration over that time interval is:

$$\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t} = \frac{(v_f - v_o)\,\hat{x}}{t_f - t_0} = \hat{x}\frac{\Delta v}{\Delta t}\,. \tag{4}$$

4c. Instantaneous Acceleration. The instantaneous acceleration, called simply "the acceleration," is the limit of the average acceleration

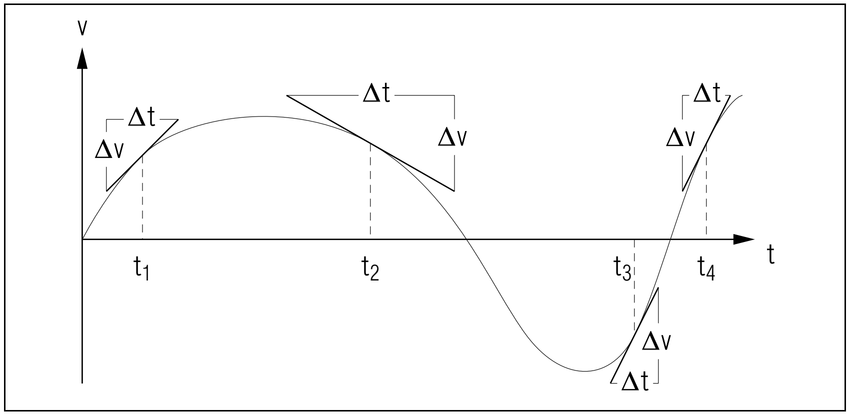


Figure 8. The small triangles show how to determine instantaneous acceleration $\Delta v/\Delta t$ at times t_1, t_2, t_3, t_4. This is not the velocity corresponding to the displacement in Fig. 4.

as $t_f \to t_0$. Dropping the unit vectors in Eq. (4) and going to the limit,

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \equiv \frac{dv}{dt} = \frac{d^2x}{dt^2}\,. \qquad (5)$$

This is "the acceleration of a particle at the time t_0." The acceleration is in the direction of the x-axis and has the dimensions length/time^2. When its value is non-zero, its direction may be to the right (positive value) or to the left (negative value).

4d. Instantaneous Acceleration From Velocity Graph. A curve of velocity versus time, whether the velocities are obtained from graphs or tables, can be quite useful. Not only does the slope give instantaneous acceleration but, as we shall see later, the area between the velocity curve and the time axis gives the displacement. The slope, dv/dt (which is also a), can be determined by drawing tangents and triangles at desired times (see Fig. 8). Here we drew the same shape for $v(t)$ as we did for $x(t)$ in Fig. 4 so as to emphasize that acceleration relates to velocity in somewhat the same manner as velocity relates to position.

4e. Instantaneous Acceleration From Position Graph. Since the slope of the velocity curve, dv/dt, is the time rate of change of velocity, it is d^2x/dt^2 which is called the "bending function" of the position/time curve. It is instructive to draw separate position, velocity and acceleration curves, one above the other, using a common time scale (see Fig. 9).

Geometrically, $a(t)$ is the rate of change of the slope of $x(t)$; it is the rate at which that function "bends." For instance, in Fig. 7 the slope is positive at $t = 0.10\,\text{s}$ but negative at $t = 0.30\,\text{s}$. In fact, the slope decreases continuously from $t = 0.10\,\text{s}$ to $t = 0.30\,\text{s}$ as the curve continues to bend negatively. Therefore the acceleration is negative throughout this interval.

In general, the acceleration a is positive where the graph of x as a function of t bends upward (positively), like an outstretched palm, as one proceeds to the right. Of course $a = 0$ where the graph is a straight line; a is negative when the curve is bending negatively downward.

Suppose, for example, we wish to examine the motion of a photophobic bug that continually moves in order to stay in the (noonday) shadow of a swinging pendulum. The bug's motion, which is technically called "simple harmonic motion" (students may question the word "simple"), can be described by the equation:

$$x = A \sin \omega t.$$

Here A is the farthest the bug gets from the center of its "back and forth" travels and ω ("omega") is 2π times the bug's number of complete circuits per unit time.

The velocity of the bug is the first derivative of position:[1]

$$v = \frac{dx}{dt} = \omega A \cos \omega t\,.$$

Its acceleration is the next derivative:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -\omega^2 A \sin \omega t\,.$$

This can be written:

$$a = -\omega^2 x\,.$$

Figure 9 shows the bug's position, velocity, and acceleration as functions of time. You should check to see if each of the lower two curves is the slope of the one above it, and that the third is the bending function of the first. Figure 10 illustrates what happens when there is constant position, velocity, and/or acceleration. This position curve is composed of several distinct segments, as can be seen more easily in the velocity and acceleration curves. Where the position curve is bending downward as

[1]See "Review of Mathematical Skills - Calculus: Differentiation and Integration" (MISN-0-1).

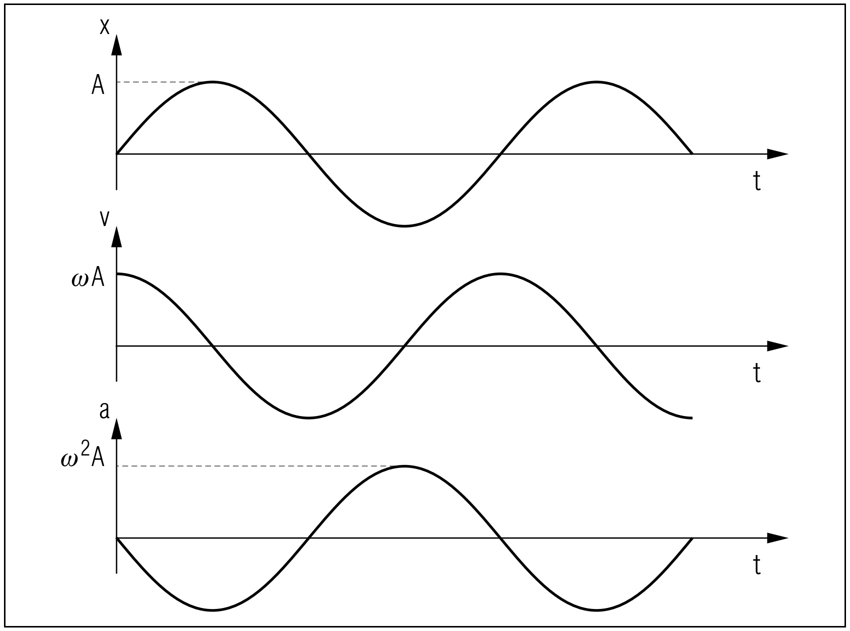


Figure 9. Plots of bug position, velocity and acceleration on the same time scale (see text).

time increases, note that the velocity is decreasing and the acceleration is negative. Where the position curve is bending upward as time increases, note that the velocity is increasing and the acceleration is positive. The acceleration is zero at the point of inflection, the point where the bending changes from downward to upward and the acceleration from negative to positive. The acceleration is also zero wherever a straight line segment of the position curve shows that the velocity is constant.

Note the difference in appearance between the curves of Fig. 9 and the three successive parabolas on the right hand side of the position curve in Fig. 10.[2] Although the displacement curves are rather similar, the graphs of velocity and acceleration are not, as can be easily seen by evaluating the derivatives. This illustrates the difficulty in accurately determining position, velocity, and acceleration relationships from graphs.

[2]The dashed lines show where a quantity is undefined (ambiguous). Where the velocity "is" a vertical line, the acceleration would be infinite. Such a situation cannot occur in real life, so such an x(t) is said to be "unphysical." Nevertheless, such x(t) curves are often close enough to real-life curves so they can be used as approximations: they are often easy to deal with mathematically.

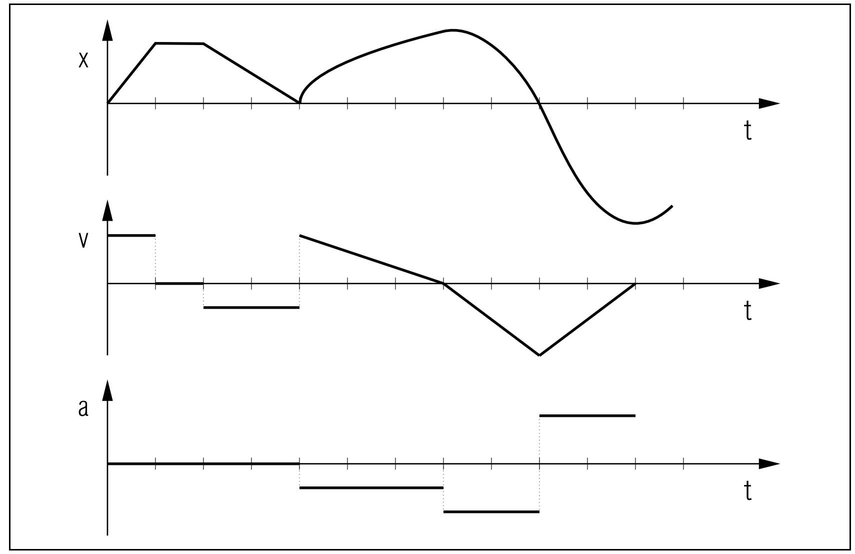


Figure 10. Concurrent plots of position, velocity and acceleration when one or more remains constant with time.

4f. Higher Order Derivatives. Derivatives of position beyond the second can be taken and in general they will be non-zero. For example, the first derivative of acceleration, which is the third derivative of position, is called the "jiggle" or "jerk," and it is used in studying vibrations. In general, one or more of the higher derivatives is of interest only when it is directly related to some other quantity involved in the motion.

4g. Units. One of the most common accelerations is that due to gravity near the surface of the earth. Generally called "g," this is $9.8\,\mathrm{m/s^2}$, or $32\,\mathrm{ft/s^2}$. One SI unit of acceleration, therefore, is about one tenth the acceleration of gravity near the surface of the earth. When dropped from rest near the surface of the earth, a particle undergoes an increase in velocity of about $1\,\mathrm{m/s}$ every tenth of a second. Half way to the moon (a distance of 30 earth radii, or $2 \times 10^7\,\mathrm{m}$), the acceleration of gravity is about one SI unit, $1\,\mathrm{m/s^2}$. A particle in that vicinity and in free fall would find its velocity increasing toward the earth at the rate of $1\,\mathrm{m/s}$ every second.

4h. Example. A Problem: Given that a particle moves along the x-axis with acceleration $a(t) = A + Bt^2$, starting from rest at $x = 5.0\,\mathrm{m}$ at $t = 0$. Find its position at all instants of time, $x(t)$.

Solution: Since a = dv/dt, write:[1]

$$v = \int dv = \int a\,dt = \int [A + Bt^2]\,dt = A\int dt + B\int t^2\,dt$$
$$= At + \frac{1}{3}Bt^3 + C\,,$$

where C is a constant that can be determined from the given initial condition that $v = 0$ when $t = 0$; $v(0) = 0$. To do so, we can set $t = 0$ in the equation above to obtain:

$$0 = 0 + 0 + C\,,$$

so

$$v(t) = At + Bt^3/3\,.$$

Next use $v = dx/dt$ to obtain:

$$x = \int dx = \int v\,dt = \int [At + \frac{1}{3}Bt^3]\,dt = \frac{1}{2}At^2 + \frac{1}{12}Bt^4 + D\,,$$

and applying the initial conditions on x we get:

$$x(t) = \frac{1}{2}At^2 + \frac{1}{12}Bt^4 + 5.0\,\mathrm{m}\,.$$

5. $a(t) \to v(t) \to x(t)$ Using Integration

5a. Start With Acceleration. In dynamics it is common to analyze the motion of an object by examining its acceleration. This is because acceleration can often be deduced from known forces, but also because instruments that measure acceleration (“accelerometers”) are used on ships, submarines, aircraft, and rockets for “inertial navigation.” Accelerometers are used because they need not be in contact with the earth. Assuming the acceleration has been obtained as a function of time during a journey, either by instrument or from known forces, the velocity and position of the traveler can be obtained provided they are known for some one time in the journey (for example, at the beginning point).

5b. Change in Velocity From Acceleration Graph. The area between an acceleration curve and the time axis is the integral $\int a(t)\,dt$, so this gives the change in velocity over the period of time being used.

The sign of the area gives the sign of the acceleration, hence determines the acceleration's direction and this can be either positive or negative. Therefore the total or net change in velocity over any period of time is equal to the net area that is bounded by the beginning and ending times (see Fig. 11). The average acceleration for the interval is the change in velocity during the time interval, the net area, divided by the length of the time interval.

5c. Velocity as an Integral. Starting with the defining equation for acceleration, $a(t) = dv(t)/dt$, we change the symbol for time from t to t' and then integrate both sides of the equation with respect to t':

$$\int_{t_0}^{t} \left(\frac{dv}{dt'}\right) dt' = \int_{t_0}^{t} a(t')\,dt'\,.$$

But:

$$\int_{t_0}^{t} \left(\frac{dv}{dt'}\right) dt' = \int_{t_0}^{t} dv = v(t) - v(t_0) \equiv v - v_0\,.$$

Then:

$$v - v_0 = \int_{t_0}^{t} a(t')\,dt'\,.$$

Rearranging,

$$v = v_0 + \int_{t_0}^{t} a(t')\,dt'\,. \qquad (6)$$

We can think of “$a(t')\,dt'$” as representing the change in velocity over the small time increment dt'. Then we can think of summing over all such small changes in velocity made during each of many small time increments in our interval from t_0 to t. The integral is then the limit as the size of each time increment approaches zero so the number of such increments in our time interval goes to infinity.

5d. Displacement From Velocity Graph. The net area between the $v(t)$ curve and the time-axis is the integral $\int v(t)\,dt$, and this is the displacement, the change in position during the period concerned (see Fig. 12).

The average velocity for the interval is the change in displacement, the net area, divided by the length of the time interval.

5e. Position as an Integral. Writing $v(t) = dx(t)/dt$ in the form $dx(t') = v(t')\,dt'$ and integrating, we get:

$$\int_{x_0}^{x} dx' = \int_{t_0}^{t} v(t')\,dt'\,.$$

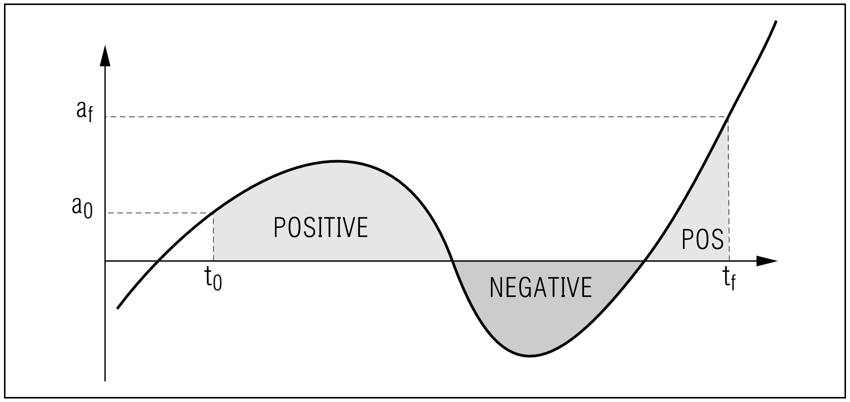


Figure 11. Graph of a hypothetical $a(t)$. The net area between the curve and the time axis gives the object's change in velocity from time t_0 to time t_f.

Integrating the left hand side, we get:

$$x(t) = x_0 + \int_{t_0}^{t} v(t')\,dt'\,, \tag{7}$$

where $v(t')dt'$ can be thought of as the small displacement of the particle in the small increment of time dt'(see Fig. 12). We can think of the integral

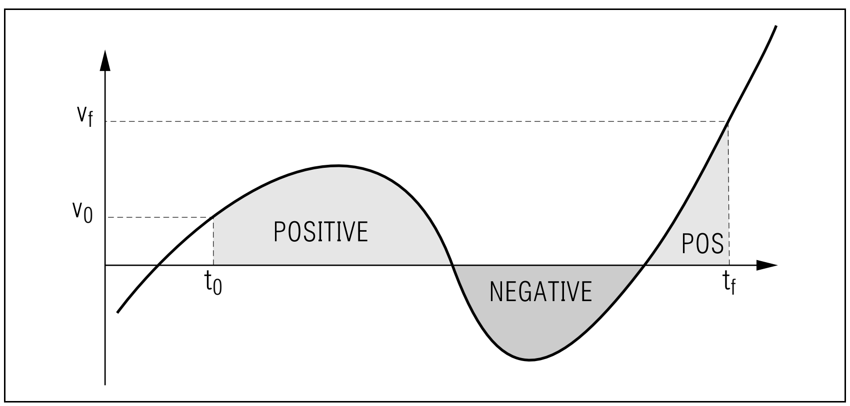


Figure 12. Graph of a hypothetical $v(t)$. The net area between the curve and the time axis gives the displacement from t_0 to t_f. The curve is not the $v(t)$ corresponding to the $a(t)$ of Fig. 11.

as the sum of many small changes in displacement.

6. Constant Acceleration

In this section we will particularize the equations of motion to the restricted case of objects undergoing constant acceleration. Such constant acceleration occurs when the net force acting on an object is itself constant in time. A number of real-life motions are close enough to this situation so that the constant acceleration equations we develop can be used as good approximations. The chief merit in using constant-acceleration equations is their mathematical simplicity.

Starting with Eq. (6) and with $a(t') = a$, a constant, we get:

$$v = v_0 + at\,. \tag{8}$$

Note that we have chosen $t_0 = 0$. Substituting that result into Eq. (7) we get:

$$\begin{aligned} x(t) &= x_0 + \int_0^t (v_0 + at')\,dt' \\ &= x_0 + v_0 \int_0^t dt' + a \int_0^t t'\,dt' \\ &= x_0 + v_0 t + \frac{1}{2}at^2\,. \end{aligned} \tag{9}$$

If v_0 is not given in a constant-acceleration problem, you can eliminate it between Eqs. (8) and (9). Try it now and make sure you get: *Help: [S-1]*

$$x = x_0 + vt - \frac{1}{2}at^2\,. \tag{10}$$

Do not memorize that equation: just make sure you can derive it when you need it.

Similarly, if t is not given you can eliminate it between Eqs. (8) and (9). Try it now and make sure you get: *Help: [S-1]*

$$v^2 - v_0^2 = 2\,a\,(x - x_0)\,. \tag{11}$$

Remember, whenever you see a, rather than a(t), as in the equations of this section, it means that the equations you are looking at are valid only for problems involving constant acceleration. If the acceleration is not constant, do not use them: instead, use equations involving $a(t)$.

Acknowledgments

This module is based, in part, on modules prepared by D. W. Joseph, J. S. Kovacs, and P. S. Signell. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

A. Communicating Word-Problem Solutions

In order for you to communicate the fact that you have solved a word-problem and have understood your solution, we have found from experience that the most effective lay-out is the one which is commonly used for communication in the professional scientific and engineering journals. We introduce you to a slight variation here as we give one more example.

Example:

Given: $x(1.0\,\mathrm{s}) = 7.0\,\mathrm{m}$;
$v(t) = \alpha t^2 + \beta t + \gamma; \alpha = 9.0\,\mathrm{m/s^3}$;
$\beta = 4.0\,\mathrm{m/s^2}; \gamma = -8.0\,\mathrm{m/s}$.

Find: $a(t)$ for $t = 2.0\,\mathrm{s}$ and $t = 4.0\,\mathrm{s}$, and $x(t)$.

a. $a(t) = \dfrac{dv(t)}{dt} = 2\alpha t + \beta$

$$\begin{aligned} a(2.0\,\mathrm{s}) &= (2)(9.0\,\mathrm{m/s^3})(2.0\,\mathrm{s}) + (4.0\,\mathrm{m/s^2}) \\ &= 36.0\,\mathrm{m/s^2} + 4.0\,\mathrm{m/s^2} \\ &= 40.0\,\mathrm{m/s^2} \end{aligned}$$

b. $a(4.0\,\mathrm{s}) = (2)(9.0\,\mathrm{m/s^3})(4.0\,\mathrm{s}) + (4.0\,\mathrm{m/s^2}) = 76.0\,\mathrm{m/s^2}$

c. $x(t) = \int v(t)\,dt = \dfrac{\alpha t^3}{3} + \dfrac{\beta t^2}{2} + \gamma t + C.$

This can be written in a more interesting manner by noting that the position at $t = 0$ is $x(0) = C$:

$$x(t) = x(0) + \frac{\alpha t^3}{3} + \frac{\beta t^2}{2} + \gamma t.$$

d. $x(t) = x(0) + \dfrac{\alpha t^3}{3} + \dfrac{\beta t^2}{2} + \gamma t$

$$7.0\,\mathrm{m} = x(0) + \frac{(9.0\,\mathrm{m/s^3})(1.0\,\mathrm{s^3})}{3} + \frac{(4.0\,\mathrm{m/s^2})(1.0\,\mathrm{s^2})}{2} + (-8.0\,\mathrm{m/s})(1.0\,\mathrm{s})$$

$$x(0) = 7.0\,\mathrm{m} - 3.0\,\mathrm{m} - 2.0\,\mathrm{m} + 8.0\,\mathrm{m} = 10.0\,\mathrm{m}.$$

$$x(t) = 10\,\mathrm{m} + \frac{\alpha t^3}{3} + \frac{\beta t^2}{2} + \gamma t$$

Notice that:

1. There is a vertical alignment of equality signs (=) as much as possible;
2. units, such as meters and seconds, are written in explicitly and their appropriate powers are computed algebraically;
3. symbolic answers are obtained first and are boxed, then numerical answers are obtained and boxed (the substitution of numbers for symbols being clearly shown); and
4. there is no extraneous material.

How did the above example shown above come to look so neat? The solution was first written out on scratch paper with false starts, erasures, crossed out parts, and other extraneous material. The pertinent parts were then arranged on this sheet in the form shown.

PROBLEM SUPPLEMENT

Note: Problems 14-17 are also on this module's *Model Exam*.

1. A particle moving along a straight line has the following positions at the indicated times:

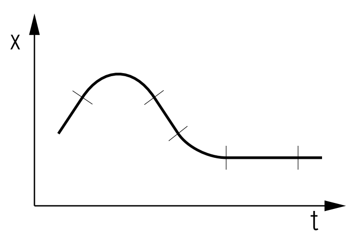
t(in s)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
x(in m)	5.2	5.5	5.9	6.4	7.0	7.7	8.5	9.4	10.4

   a. Use the table to determine the average velocity:
      i. for the interval $t = 0\,\mathrm{s}$ to $t = 0.5\,\mathrm{s}$,
      ii. for the interval $t = 0.5\,\mathrm{s}$ to $t = 0.8\,\mathrm{s}$,
      iii. for the interval $t = 0\,\mathrm{s}$ to $t = 0.8\,\mathrm{s}$.

   b. Determine the approximate instantaneous velocity from the x vs t curve:
      i. at $t = 0.4\,\mathrm{s}$,
      ii. at $t = 0.5\,\mathrm{s}$.

   c. Does the instantaneous velocity become equal to the average velocity at the midpoint of displacement or the midpoint of time? Why?

   d. Indicate how to determine the above velocities on a position-time graph.

2. 

   a. A graph of x vs t for a particle in straight-line motion is shown in the sketch. For each interval, indicate (above the curve) whether the average velocity v_{av} is $+$, $-$, or 0, and (below the curve) whether the acceleration a_x is $+$, $-$, or 0.

   b. Locate all points on the graph where the instantaneous velocity is zero.

3. The position of an object moving in a straight line is given by $x = A + Bt + Ct^2$, where $A = 1.0\,\mathrm{m}$, $B = 2.0\,\mathrm{m/s}$ and $C = -3.0\,\mathrm{m/s^2}$.

   a. What is its average velocity for the interval from $t = 0$ to $t = 2.0\,\mathrm{s}$?

   b. What are its (instantaneous) velocities at $t = 0$ and $t = 2.0\,\mathrm{s}$?

   c. What is its acceleration at each of these times?

4. A rocket is fired vertically, and ascends with a constant vertical acceleration of $+20.0\,\mathrm{m/s^2}$ for 80.0 s. Its fuel is then all used and it continues with an acceleration $g = -9.8\,\mathrm{m/s^2}$. Air resistance can be neglected.

   a. What is its altitude 80.0 s after launching?

   b. How long does it take to reach its maximum altitude?

   c. What is this maximum altitude?

5. A particle moves along the x-axis with acceleration $a(t) = A + Bt^2$, starting from rest at $x = 5.0\,\mathrm{m}$ and $t = 0$. Find its position $x(t)$.

6. You have leveled an air track and then placed a block under one end of the track. Using photocell gates and a timer, you find the length of time t it takes a glider on the track to move some convenient distance $x - x_0$. Determine the acceleration a_x of the glider from these data: $x - x_0 = 100.0\,\mathrm{cm}$, $t = 4.053\,\mathrm{s}$.

7. Water drips from a shower nozzle onto the floor 72 inches below. Neglect air resistance.

   a. How fast are the drops falling when they strike the floor?

   b. How long does it take a drop to fall?

8. A lifeguard is standing on the edge of a swimming pool when she drops her whistle. The whistle falls 4.0 ft from her hand to the water. It then sinks to the bottom of the pool at the same constant velocity with which it struck the water. It takes a total of 1.0 s to go from hand to bottom.

   a. How long was it falling through the air?

   b. How long was it falling through the water?

   c. With what velocity did it strike the water?

   d. How deep was the pool?

9. A truck traveling at 60.0 mph (88 ft/s) passes a car pulling out of a gas station. The driver of the car instantaneously steps on the gas and accelerates at 8.0 ft/s^2 and catches the truck in 0.200 mi (1056 ft). How fast was the car traveling when the truck passed it and how long did it take to catch the truck?

10. In a certain amusement park, a bell will ring when struck from below by a weight traveling upward at 10.0 ft/s. How fast must a weight be projected upward to ring a bell which is 36 feet above the ground? How long does it take to hit the bell?

11. Suppose that after many years of patient waiting, a radar tracking station was able to track an unidentified flying object (UFO). Initially the UFO was at rest, but as soon as it was sighted it started to move away from the station in a straight line. Its speed along this line was measured to be $v = \alpha t - \beta t^3$ where $\alpha = 300\,\text{mi/s}^2$ and $\beta = 0.75\,\text{mi/s}^4$ during the time it was observed, until it disappeared 20 s after first sighting.

    a. How fast was the UFO going when it disappeared?

    b. What was its acceleration when it first started to move?

    c. How far did the UFO go during the 20 s?

12. A particular lightning flash is seen 5.0 s before the thunder is heard. How far away is the thunderstorm?

13. A cyclist accidentally drops a padlock off the side of a high bridge. One second later he disgustedly throws the key downwards at 12 m/s after it. Does the key overtake the padlock? If so, when and where?

14. The position of a particle is given by: $x = A - Bt + Dt^3 - Et^4$.

    a. Find the velocity.

    b. Find the acceleration.

    c. Find average velocity for the interval $t = 0$ to $t = 3$ s.

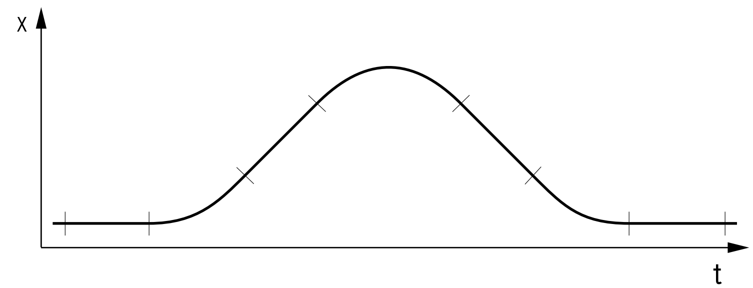
15. A physics professor at the football stadium drives two miles home at 30 mph to get her football tickets, discovers them in her purse, and immediately drives back at 20 mph because the traffic is worse. What was her average velocity for the round trip?

16. A salesman brags that a car will accelerate from 10 mi/hr (4.47 m/s) to 75 mi/hr (33.5 m/s) in 12 s.

a. Find the average acceleration in m/s^2 during this time interval (do not assume constant acceleration).

b. Assuming constant acceleration, find the distance and time at which the car would attain the speed of 55 mi/hr (24.6 m/s), starting from 10 mi/hr.

17. a. A graph of x vs t for a particle in straight line motion is shown in the sketch.



For each interval between the hash marks:

i. mark, above the curve, whether the average velocity v_{av} is +, −, or 0; and,

ii. mark, below the curve, whether the acceleration a is +, −, or 0.

b. Identify all points on the graph where the instantaneous velocity is zero.

Brief Answers:

1. a. $v_{av(0-5)} = \dfrac{x_5 - x_0}{t_5 - t_0} = \dfrac{7.7\,\text{m} - 5.2\,\text{m}}{0.5\,\text{s}} = \dfrac{2.5\,\text{m}}{0.5\,\text{s}} = 5.0\,\text{m/s}$

$v_{av(5-8)} = \dfrac{x_8 - x_5}{t_8 - t_5} = \dfrac{10.4\,\text{m} - 7.7\,\text{m}}{0.8\,\text{s} - 0.5\,\text{s}} = \dfrac{2.7\,\text{m}}{0.3\,\text{s}} = 9.0\,\text{m/s}.$

$v_{av(0-8)} = \dfrac{x_8 - x_0}{t_8 - t_0} = \dfrac{10.4\,\text{m} - 5.2\,\text{m}}{0.8\,\text{s}} = \dfrac{5.2\,\text{m}}{0.8\,\text{s}} = 6.5\,\text{m/s}$

b. $v_4 = \dfrac{x_5 - x_3}{t_5 - t_3} = \dfrac{7.7\,\text{m} - 6.4\,\text{m}}{0.5\,\text{s} - 0.3\,\text{s}} = \dfrac{1.3\,\text{m}}{0.2\,\text{s}} = 6.5\,\text{m/s}$

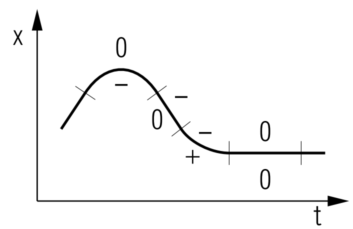
$v_5 = \dfrac{x_6 - x_4}{t_6 - t_4} = \dfrac{8.5\,\text{m} - 7.0\,\text{m}}{0.6\,\text{s} - 0.4\,\text{s}} = \dfrac{1.5\,\text{m}}{0.2\,\text{s}} = 7.5\,\text{m/s}$

Note: You might have chosen different time intervals.

c. Inspection of the table shows that the instantaneous velocity, as indicated by the increases in displacement in each time interval, is increasing uniformly, indicating that the acceleration is constant. Combining $v = v_0 + at$ and $v_{av} = (v + v_0)/2$, we get $v_{av} = v_0 + (a\,t/2)$ which shows that the average velocity occurs at time $t/2$, as shown by the calculations above.

d. See Figs. P1a-c.

2. a. See the figure.



b. Highest point and lowest segment of the curve.

3. a. $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{(A + Bt_2 + Ct_2^2) - (A + Bt_1 + Ct_1^2)}{t_2 - t_1}$

At $t_1 = 0$, $t_2 = 2\,\text{s}$:

$$v_{av} = \frac{(A + Bt_2 + Ct_2^2) - (A)}{t_2} = \frac{t_2(B + Ct_2)}{t_2} = B + Ct_2$$

$$= (2.0\,\text{m/s}) + (-3.0\,\text{m/s}^2)(2.0\,\text{s}) = 2.0\,\text{m/s} - 6.0\,\text{m/s}$$

$$= -4.0\,\text{m/s}.$$

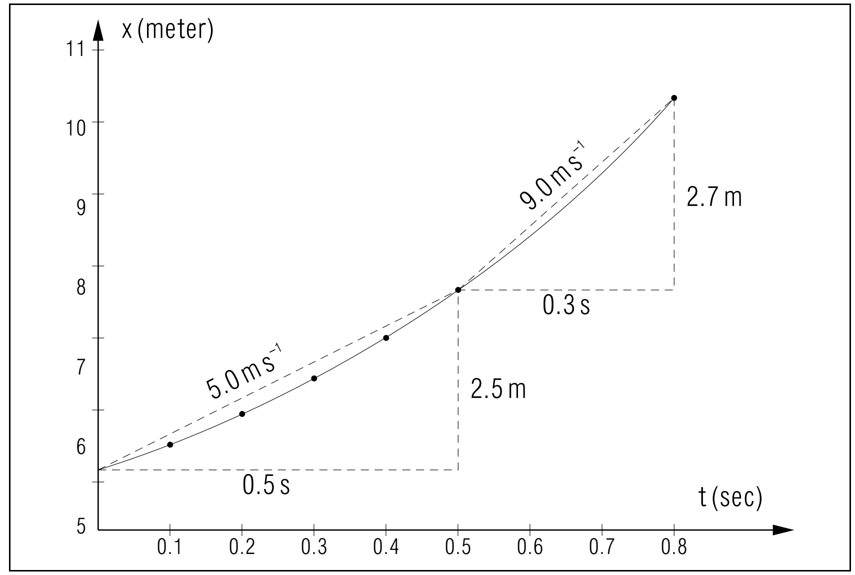


Figure 13. The triangles show how to calculate the average velocities for the intervals $t_0 - t_5$ and $t_5 - t_8$.

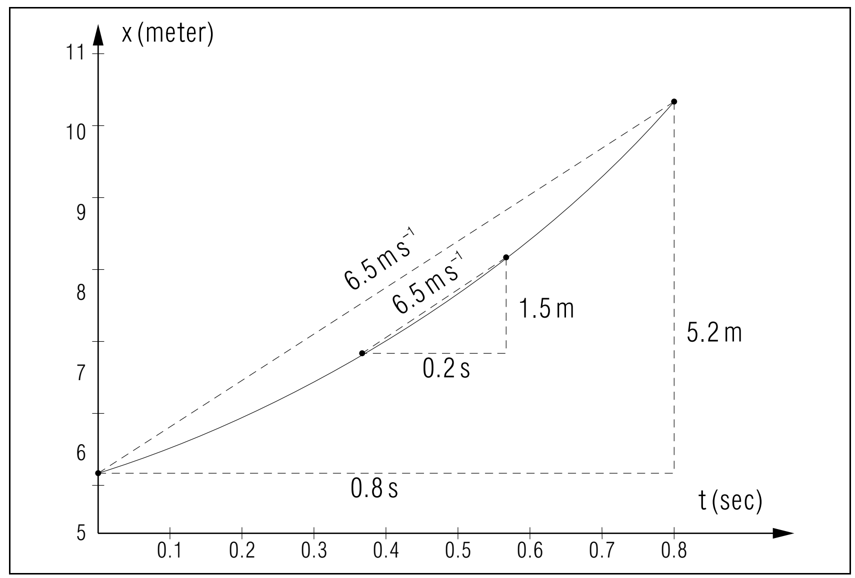


Figure 14. The triangles show that the average velocity for the interval $t_0 - t_8$ equals the instantaneous velocity at t_4.

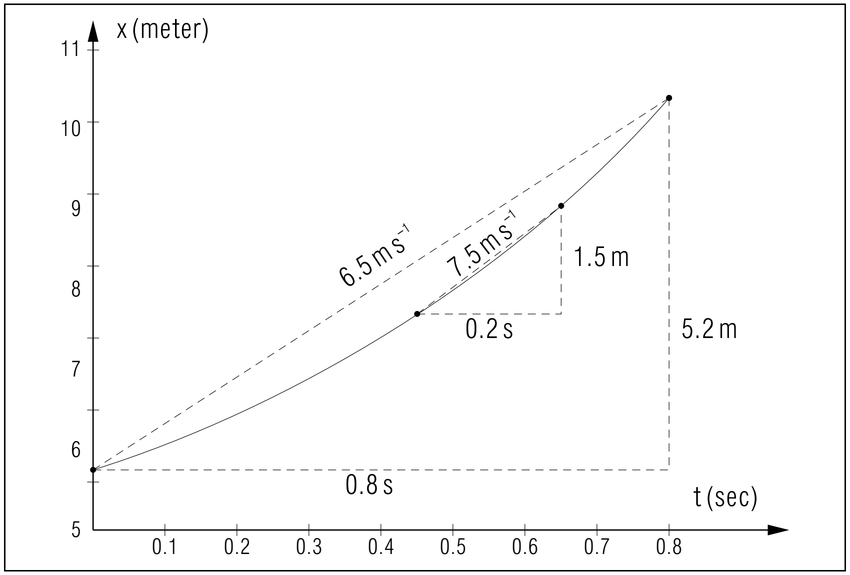


Figure 15. The triangles show that the instantaneous velocity at t_5 (the approximate midpoint of displacement) does not equal the average velocity for the interval $t_0 - t_8$.

b. $v(t) = \dfrac{dx(t)}{dt} = \dfrac{d}{dt}(A + Bt + Ct^2) = B + 2Ct$

$v(0) = B = 2.0\,\mathrm{m/s}$

$v(2.0\,\mathrm{s}) = 2.0\,\mathrm{m/s} + 2(-3.0\,\mathrm{m/s^2})(2.0\,\mathrm{s}) = -10\,\mathrm{m/s}.$

c. $a(t) = \dfrac{dv(t)}{dt} = \dfrac{d}{dt}(B + 2Ct) = 2C = -6.0\,\mathrm{m/s^2}$ at t_1 and t_2.

4. Given $v_0 = 0$, $a = +20\,\mathrm{m/s^2}$, $t = 80\,\mathrm{s}$, $g = -10\,\mathrm{m/s^2}$,

a. $x_1 = v_0 t + \dfrac{1}{2}at^2$

$= 0 + \dfrac{1}{2}(20\,\mathrm{m/s^2})(80\,\mathrm{s})^2 = 6.4 \times 10^4\,\mathrm{m}$

$v = v_0 + at = 0 + (20\,\mathrm{m/s^2})(80\,\mathrm{s}) = 1600\,\mathrm{m/s}.$

The rocket continues upward until it stops;

$t = \dfrac{v - v_0}{g} = \dfrac{0 - 1600\,\mathrm{m/s}}{-10\,\mathrm{m/s^2}} = 160\,\mathrm{s}.$

b. Total time to rise $= 80\,\mathrm{s} + 160\,\mathrm{s} = 2.4 \times 10^2\,\mathrm{s}.$

Distance upward after burnout:

$x = v_b t + \dfrac{1}{2}gt^2$

$= (1600\,\mathrm{m/s})(160\,\mathrm{s}) + \dfrac{1}{2}(-10\,\mathrm{m/s^2})(160\,\mathrm{s})^2 = 128,000\,\mathrm{m}.$

Alternatively,

$v^2 - v_0^2 = 2\,a\,x$

$x = \dfrac{-v_0^2}{2a} = \dfrac{-(1600\,\mathrm{m/s})^2}{2(-10\,\mathrm{m/s^2})} = 128,000\,\mathrm{m}.$

c. Maximum altitude $= 64,000\,\mathrm{m} + 128,000\,\mathrm{m} = 1.92 \times 10^5\,\mathrm{m}$

5. Since $a_x = \dfrac{dv_x}{dt}$, you can write

$v_x = \int dv_x = \int a_x dt = \int (A + Bt^2)dt = A\int dt + B\int t^2 dt$

$= At + \dfrac{1}{3}Bt^3 + C.$

Set $t = 0$ to obtain $0 = v_x(0) = C$.

Next,

$x = \int dx = \int v_x dt = \int (At + \dfrac{1}{3}Bt^3)dt = \dfrac{1}{2}At^2 + \dfrac{1}{12}Bt^4 + D$

or:

$x(t) = \dfrac{1}{2}At^2 + \dfrac{1}{12}Bt^4 + D.$

This time, the initial condition tells us that $D = 5.0\,\mathrm{m}$; so the final expression is

$$x(t) = \frac{1}{2}At^2 + \frac{1}{12}Bt^4 + 5.0\,\text{m}.$$

6. $v_{av} = \dfrac{x - x_0}{t} = \dfrac{100.0\,\text{cm}}{4.053\,\text{s}} = 24.67\,\text{cm/s}.$

   $$x - x_0 = v_0 t + \frac{1}{2}at^2$$

   $$a = [(x - x_0) - v_0 t]\,2/t^2.$$

   If we set $v_0 = 0$ at $t = 0$,

   $$a = \frac{2(x - x_0)}{t^2} = \frac{2(100.0\,\text{cm})}{(4.053\,\text{s})^2} = 12.18\,\text{cm/s}^2.$$

   If $v_0 > 0$ at $t = 0$, $a < 12.18\,\text{cm/s}^2$.

7. Since this problem is one-dimensional, it is convenient to take the direction for the positively-increasing x-axis as downward. Then $x = 72\,\text{in} = 6\,\text{ft}$, $v_0 = 0$ at $t = 0$, $a = g = 32\,\text{ft/s}^2$,

   a. $v^2 - v_0^2 = 2\,a\,x$

      $$v = (2\,a\,x)^{1/2} = \left[(2)(32\,\text{ft/s}^2)(6\,\text{ft})\right]^{1/2} = 20\,\text{ft/s}.$$

   b. $v - v_0 = at$

      $$t = \frac{v}{a} = \frac{20\,\text{ft/s}}{32\,\text{ft/s}^2} = 0.62\,\text{s}.$$

      Alternatively,

      $$x = v_0 t + \frac{1}{2}at^2$$

      $t = (2x/a)^{1/2} = \left[\dfrac{(2)(6\,\text{ft})}{32\,\text{ft/s}^2}\right]^{1/2} = 0.615\,\text{s}$. The difference in time results from rounding error.

8. a. $x(t) = x(0) + v(0)t + \dfrac{1}{2}at^2.$

      We orient the x-axis to increase positively downward so $a = g$. We put $t = 0$ at the instant of drop so $v(0) = 0$ and we put the origin at the hand so $x(0) = 0$. Let $d_a \equiv$ distance through air; by (1) it is:

      $$d_a = \frac{1}{2}gt_a^2,$$

   where $t_a \equiv$ time through air.

   $$t_a = (2d_a/g)^{1/2} = \left[2(4.0\,\text{ft})/(32\,\text{ft/s}^2)\right]^{1/2} = \left(\frac{1}{4}\,\text{s}^2\right)^{1/2} = \frac{1}{2}\,\text{s}.$$

   b. Let $t_w \equiv$ time in water, t_t = total time from hand to bottom $= t_w + t_a$

      $$t_w = t_t - t_a = 1.0\,\text{s} - \frac{1}{2}\,\text{s} = \frac{1}{2}\,\text{s}.$$

   c. Velocity at water $\equiv v_w = v(t_a) = gt_a = (32\,\text{ft/s}^2)\left(\dfrac{1}{2}\,\text{s}\right) = 16\,\text{ft/s}.$

   d. Let $d_w \equiv$ distance in water $= v_w t_w = (16\,\text{ft/s})\left(\dfrac{1}{2}\,\text{s}\right) = 8\,\text{ft}.$

9. Time $= \dfrac{\text{distance traveled}}{\text{velocity of truck}} = \dfrac{1056\,\text{ft}}{88\,\text{ft/s}} = 12\,\text{s}.$

   For the car, $x = v_0 t + \dfrac{1}{2}at^2$,

   $$v_0 = (x - \frac{1}{2}at^2)/t$$

   $$= x/t - \frac{1}{2}at = \frac{1056\,\text{ft}}{12\,\text{s}} - \frac{1}{2}\left(8\,\text{ft/s}^2\right)(12\,\text{s}) = 40\,\text{ft/s}.$$

10. Take $x = 36\,\text{ft}$, $v = 10\,\text{ft/s}$, $a = g = -32\,\text{ft/s}^2$.

    $$v^2 - v_0^2 = 2ax$$

    $$v_0 = (v^2 - 2ax)^{1/2} = \left[(10\,\text{ft/s})^2 - 2(-32\,\text{ft/s}^2)(36\,\text{ft})\right]^{1/2},$$

    $$= (2400\,\text{ft}^2/\,\text{s}^2)^{1/2} = 49\,\text{ft/s}$$

    $$t = \frac{v - v_0}{a} = \frac{10\,\text{ft/s} - 49\,\text{ft/s}}{-32\,\text{ft/s}^2} = 1.22\,\text{s}.$$

    Checking:

    $$x = v_0 t + \frac{1}{2}at^2$$

    $$= (49\,\text{ft/s})(1.22\,\text{s}) + \frac{1}{2}(-32\,\text{ft/s}^2)(1.22\,\text{s})^2 = 36.0\,\text{ft}.$$

It is often convenient to carry an extra significant figure in calculations.

11. $v(t) = \alpha t - \beta t^3$; $\alpha = 300\,\text{mi/s}^2$, $\beta = \dfrac{3}{4}\,\text{mi/s}^4$.

   a. $v(20\,\text{s}) = (300\,\text{mi/s}^2)(20\,\text{s}) - (\frac{3}{4}\,\text{mi/s}^4)(20\,\text{s})^3$

   $= 6.0 \times 10^3\,\text{mi/s} - 6.0 \times 10^3\,\text{mi/s} = 0.$

   b. $a(t) = \dfrac{dv(t)}{dt} = \alpha - 3\beta t^2$.

   When the object "first started to move" probably means $t = 0$ since that is the first time when $v = 0$. The acceleration at that time was:

   $a(0) = \alpha = 300\,\text{mi/s}^2$.

   c. $x(t) = \int(\alpha t - \beta t^3)dt = \dfrac{\alpha t^2}{2} - \dfrac{\beta t^4}{4} + C$. $x(0) = C$, hence $x(t) - x(0) = \dfrac{\alpha t^2}{2} - \dfrac{\beta t^4}{4}$. Now let $d(t) \equiv$ distance traveled since $t = 0$, which is also the distance traveled since $v = 0$. Then:

   $d(t) = x(t) - x(0) = \dfrac{\alpha t^2}{2} - \dfrac{\beta t^4}{4}$,

   $$d(20\,\text{s}) = \frac{(300\,\text{mi/s}^2)(20\,\text{s})^2}{2} - \frac{\left(\frac{3}{4}\,\text{mi/s}^4\right)(20\,\text{s})^4}{4}$$
   $$= 6.0 \times 10^4\,\text{mi} - 3.0 \times 10^4\,\text{mi} = 3.0 \times 10^4\,\text{mi}.$$

12. Velocity of light $= 3.0 \times 10^8\,\text{m/s}$.

   Velocity of sound $= 3.3 \times 10^2\,\text{m/s}$.

   We may neglect the time it takes for light to reach us.

   $x = vt = (3.3 \times 10^2\,\text{m/s})(5.0\,\text{s}) = 1.7 \times 10^3\,\text{m}$.

13. $x(t) = x(0) + v(0)t + at^2/2$.

   We orient $x(t)$ downward, so $a = g = 9.8\,\text{m/s}^2$.

   For the padlock, $x = gt^2/2$.

   For the key, $v_0 = 12\,\text{m/s}$, $x = v_0(t - 1.0\,\text{s}) + \dfrac{1}{2}g(t - 1.0\,\text{s})^2$.

   Solving simultaneously, $\dfrac{1}{2}gt^2 = v_0(t - 1.0\,\text{s}) + \dfrac{1}{2}g(t - 1.0\,\text{s})^2$

   When: $t = 3.2\,\text{s}$ after dropping the padlock.

   Where: $x = 51\,\text{m}$.

14. a. $v = -B + 3Dt^2 - 4Et^3$.

   b. $a = 6Dt - 12Et^2$.

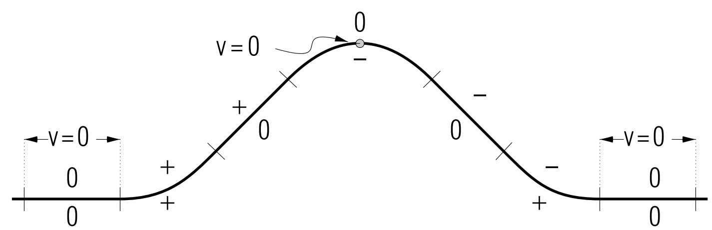
   c. $v_{av} = -B + 9s^2D - 27s^3E$.

15. Average Velocity $= v_{av} = \dfrac{x(t_{final}) - x(t_{initial})}{t_{final} - t_{initial}} = 0$. Note tht the average *speed* is not zero.

16. a. $a = 2.42\,\text{m/s}^2$, $x = 228\,\text{m}$.

   b. $x = 121\,\text{m}$, $t = 8.32\,\text{s}$.

17.



SPECIAL ASSISTANCE SUPPLEMENT

S-1 *(from TX-Sect. 6)*

The two referenced equations each contain the *symbol* that you are being asked to eliminate. Label either one of the equations #1, the other #2. Solve equation #1 for the *symbol*. You end up with:

$$symbol = some\ stuff.$$

Now, everywhere that the *symbol* occurs in equation #2 you must replace it with the *some stuff*. Because you substituted for it, the *symbol* is gone from the equation.

If necessary, solve the resulting equation for whatever is of interest.

MODEL EXAM

1. See Output Skill K1 in this module's *Problem Supplement*.
2. See Problem 14 in this module's *Problem Supplement*.
3. See Problem 17 in this module's *Problem Supplement*.
4. See Problem 15 in this module's *Problem Supplement*.
5. See Problem 16 in this module's *Problem Supplement*.