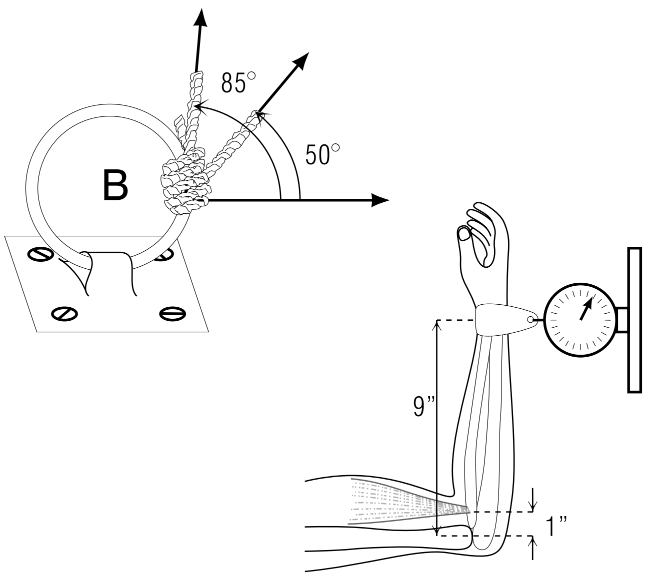
PHYSNET

MISN-0-5

TOOLS FOR STATIC EQUILIBRIUM



Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

TOOLS FOR STATIC EQUILIBRIUM

by

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Title: **Tools for Static Equilibrium**

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Input Skills:

1. Given two or more forces, find their vector sum (MISN-0-14).
2. Resolve any force into perpendicular components (MISN-0-72).
3. Evaluate the vector product of two vectors (MISN-0-2).
4. State the relationship between torque and angular acceleration (MISN-0-33).
5. State the relationship between force and acceleration specifically including contact forces (MISN-0-16).

Output Skills (Knowledge):

K1. Distinguish a set of concurrent forces from a set of nonconcurrent ones.

K2. State the equilibrium condition for an object which is acted upon by a set of concurrent forces.

Output Skills (Problem Solving):

S1. Given an object in static equilibrium and its environment, determine the forces that act on the object and draw a one-body force diagram ("free-body diagram") indicating these forces.

S2. Using the force diagram for an object, apply the conditions for static equilibrium to determine the unknown magnitudes and/or directions of forces in a set of concurrent forces.

Post-Options:

1. "Static Equilibrium; Centers of Force, Gravity, and Mass" (MISN-0-6).

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Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

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1. Force Diagrams

1a. The Importance of Force Diagrams. In simplest terms, a system in static equilibrium is one that does not move. Figure 1 shows examples of common types of systems studied in this subject.

We never begin calculations to determine unknown forces in such systems as those shown in Fig. 1 without first constructing a one-body force diagram. This gives us added assurance that we understand the system being analyzed and that we are using all the forces pertinent to our calculations. In addition, such diagrams aid us in determining whether the forces acting are concurrent or non-concurrent. The forces acting on an object are concurrent if the lines of action of the forces all meet at a common point (otherwise they are non-concurrent). In Figs. 2 and 4 dashed lines extended from the arrows indicate the lines of action. In Fig. 1a the forces are concurrent while in Fig. 4 the forces on the beam are non-concurrent. The importance of this distinction will be considered in a later section.

1b. Example: A Hanging Weight. Figure 1(a) shows a weight suspended by cords connected to point P and attached at two points A and B on the ceiling. Our force diagram for the point P is shown in Fig. 2. The three cords are the only force contacts with point P. If we know the weight being supported we can represent the tension in the vertical cord

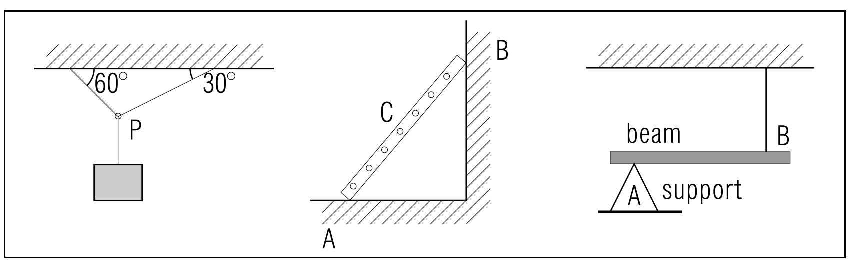

Figure 1. Typical examples of static equilibrium: (a) A weight suspended from cords; (b) A ladder resting against a wall; (c) A uniform beam held by a cord and a support.

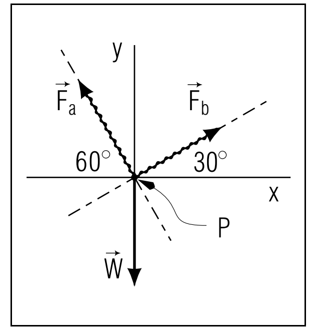

Figure 2. Force diagram for point P of Fig. 1(a).

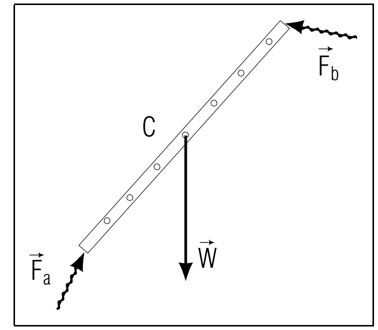

Figure 3. Force diagram of the ladder in Fig. 1(b).

by a force $\vec{W}$ acting downward on P. For the other two cords we know the directions of the forces but not the magnitudes. They are drawn as wavy arrows to indicate the known directions but unknown lengths. Note: The directions are discerned from certain properties of cords. A flexible cord can only sustain a tension, that is, it can pull on point P but not push. Furthermore, the cord will align itself in the same direction as the force which it exerts.

1c. Example: A Leaning Ladder. Figure 1(b) shows a ladder of given weight W resting against a wall. In this case we choose the ladder as the object of interest. We assume the weight of the ladder to act at its center C.[1] The other two force contacts are at the floor and the wall. Figure 3 shows our force diagram for the ladder. $\vec{F}_a$ is the force exerted on the ladder by the floor and $\vec{F}_b$ is the force exerted on the ladder by the wall. The direction and magnitude of forces $\vec{F}_a$ and $\vec{F}_b$ may be unknown. We use wavy arrows to indicate a lack of knowledge of both direction and magnitude.

1d. Example: A Balanced Beam. Figure 1(c) shows a uniform beam resting on a support near one end and suspended by a cord near the other end. We construct a force diagram for the beam. Again we assume the weight to act at the center of the beam. In this case the cord is assumed to be vertical and therefore the forces exerted by the cord and the support are both vertically upward. In Fig. 4 these two forces are shown as $\vec{F}_a$ and $\vec{F}_b$. We use wavy arrows to indicate the unknown

[1]The reasons are examined in "Static Equilibrium" (MISN-0-6).

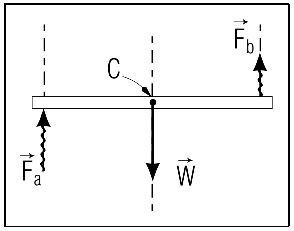

Figure 4. Force Diagram for the beam in Fig. 1(c).

magnitudes but known direction. Figure 4 shows our force diagram.

1e. Action/Reaction Forces. The concept of action-reaction forces is useful in determining the forces that act on different parts of a system. Experiment and intuition shows us that when one object exerts a force on a second object, the second object always reacts by exerting a force on the first. The forces in this "pair" always have the same magnitudes but opposite directions. For example, looking at Figs. 1(b) and 3, $\vec{F}_a$ is the force that the floor exerts on the ladder and hence the ladder pushes on the floor with a force of $-\vec{F}_a$ (same magnitude, opposite direction). This was summarized by Newton in his third law of motion: for every action there is always an equal and opposite reaction.

▷ A lamp hangs from a ceiling, supported by two strings (see Fig. 5a). The tension in string AB is 4.0 lb, while the tension in string BC is 5.35 lb. Construct the force diagram for point B.

▷ A telecommunications tower is erected (see Fig. 5b.). If the tension in the left side cable is 7500 lb, construct the force diagram for the tower. (Data from Cook Communications Inc., Lansing, MI)

▷ For more examples see this module's *Special Assistance Supplement*, Sect. 2-3.

2. Concurrent Forces

2a. Static Equilibrium for a Point. Newton stated his first law as something like this: every body persists in its state of rest or constant velocity unless it is compelled to change by forces applied to it. One interpretation of Newton's first law as applied to objects staying at rest is: the net force (the vector sum of all forces) acting on the body must be zero. In Figs. 1a and 2 we show a point P which is at rest and hence the net force must be zero. In other words, the vector sum of the forces

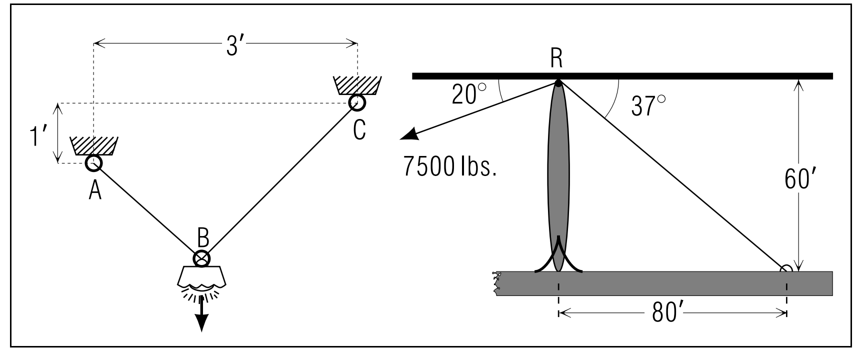

Figure 5. (a) A lamp hanging from a ceiling; (b) a telecommunications tower.

acting must be zero:

$$\vec{F}_a + \vec{F}_b + \vec{W} = 0 \tag{1}$$

Because the forces all act at the same point P, this is the only condition necessary for P to remain at rest. This is the condition for static equilibrium of a point.

2b. Concurrent Forces on an Extended Object. There is a special case of static equilibrium for extended (non-point) objects: if all the lines of force intersect at some point P, the forces act just as if they all originated at P. Even if P is geometrically not located on the object, as in Fig. 6, this statement is still true (if it makes you feel better, you can consider the object to be part of a much larger rigid structure that includes point P). Such forces are said to be "concurrent."

Based on the above, we now state this equilibrium condition: If the forces acting on a rigid object are concurrent, then the only condition necessary for equilibrium is that the vector sum of the forces is zero. In the case of the object in Fig. 6,

$$\vec{F}_a + \vec{F}_b + \vec{W} = 0\,. \tag{2}$$

is the equation for equilibrium.

In general, if a rigid object is acted upon by N concurrent forces the equilibrium condition gives us:

$$\vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_N \equiv \sum_{i=1}^{N} \vec{F}_i = 0\,. \tag{3}$$

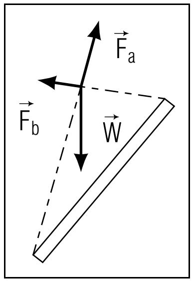

Figure 6. Forces can intersect outside the object.

2c. Cartesian Component Equations. If $\vec{F}_R = 0$, F_{Rx} and F_{Ry} must also be zero. Thus, taking components of Eq. (3) gives:

$$F_{Rx} = F_{1x} + F_{2x} + \ldots + F_{Nx} = \sum_{i=1}^{N} F_{ix} = 0\,,$$

$$F_{Ry} = F_{1y} + F_{2y} + \ldots F_{Ny} = \sum_{i=1}^{N} F_{iy} = 0\,. \tag{4}$$

The equilibrium condition, Eq. (1), in component form is:

$$F_{ax} + F_{bx} + W_x = 0\,,$$

$$F_{ay} + F_{by} + W_y = 0\,. \tag{5}$$

These equations involve components of vectors and hence there may be positive or negative values. In this case the values for F_{ax} and W_y will be negative while the others will be positive. ($W_x = 0$, as there is no component of weight in the x-direction). Equations (5) become:

$$-F_a \cos 60^\circ + F_b \cos 30^\circ = 0\,,$$

$$F_a \sin 60^\circ + F_b \sin 30^\circ - W = 0\,. \tag{6}$$

We thus have two equations and two unknowns, allowing us to solve for F_a and F_b if given W.

2d. Example: A Broken Leg. ▷ A Patient with a broken leg is in the Health Center. To allow his leg to set, a "Bucks" traction device (nothing to do with the cost involved) is used, as shown in Fig. 7. If the traction required for the bone to set is 22 lb, find the weight needed to produce this traction. (Data from the Michigan State University College of Osteopathic Medicine.) (Ans. 14 lb; *Help: [S-1]* near the end of the *Special Assistance Supplement*).

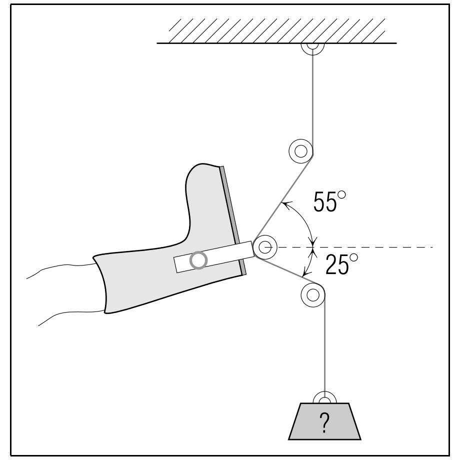

Figure 7. A traction device for setting broken bones.

▷ For more examples see this module's *Special Assistance Supplement*, Sect. 4.

2e. Example: A Ditched Car. ▷ A driver runs off the road in a snowstorm (the kind we have in April) and becomes stranded in a ditch. She knows she can't use brute strength to pull the car out, so she tries an alternate method. First, she ties a length of rope to a huge boulder, and then connects it to her car. When a 40 lb force is exerted perpendicular to the rope, the rope is displaced to an angle of 6° (see Fig. 8). What force is then being exerted on the car? (Data from Southend Total, Lansing, MI) (Ans.: 192 lb.)

3. Non-Concurrent Forces

3a. Torque for Non-Concurrent Forces. If the forces acting on a body are not concurrent, there is a possibility that the object is not in rotational equilibrium. Consider a door you are pushing open. The forces exerted by the hinges and your hand are not concurrent and the door is rotating. In contrast, the beam shown in Figs. 1 and 4 is acted upon

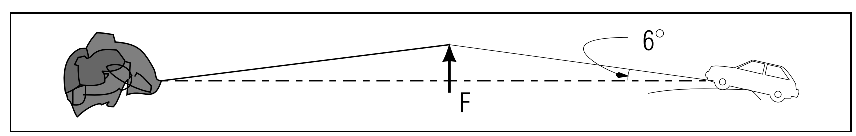

Figure 8. Illustration of force amplification.

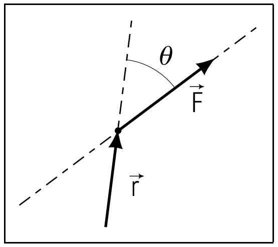

Figure 9. Force, $\vec{F}$, with given line of action and position vector $\vec{r}$.

by forces which are not concurrent and it is not rotating. Thus a rigid body which is acted upon by a set of non-concurrent forces may or may not rotate. The statement of the conditions necessary to extend static equilibrium to include non-rotation uses the concept of torque. In anticipation of the discussion of rotational equilibrium in another module,[2] we here define torque and calculate its value for simple cases.

3b. Definition of Torque. The torque, sometimes called the "moment" of a force, is defined to be (see Fig. 9):

$$\vec{\tau} = \vec{r} \times \vec{F}\,, \tag{7}$$

where $\vec{\tau}$ is the torque, $\vec{r}$ is the position vector from the "center of torque" to any point on the line of action of the force, and $\vec{F}$ is the force. Torque is a vector because it is the vector product ("cross product") of $\vec{r}$ and $\vec{F}$.[3] From the definition of the vector product, the magnitude of $\vec{r} \times \vec{F}$ is $rF\sin\theta$ and its direction is determined by the right-hand rule (into the page in this case). $F\,\sin\theta$ is just the component of the force $\vec{F}$ perpendicular to the position vector $\vec{r}$. Letting $F\,\sin\theta \equiv F_\perp$, the magnitude of the torque is just $r\,F_\perp$.

3c. Torque is Independent of Choice for $\vec{r}$. The vector $\vec{r}$ that is used to find a torque is not unique. Figure 10 shows only three of an infinite number of $\vec{r}$ vectors which could be drawn from point O to the line of action of the force $\vec{F}$. Based on this diagram and the definition of torque, Eq. (7), we get:

$$\vec{\tau} = \vec{r}_1 \times \vec{F} = \vec{r}_2 \times \vec{F} = \vec{r}_3 \times \vec{F}\,. \tag{8}$$

This $\vec{\tau}$ is directed into the page and has magnitude:

$$\tau = r_1\,F\sin\theta_1 = r_2\,F\sin\theta_2 = r_3\,F\sin\theta_3\,. \tag{9}$$

[2] "Static Equilibrium, Center of Mass" (MISN-0-6).

[3] See "Vectors: Sums, Differences, and Products" (MISN-0-2). Actually, torque is a psuedovector, which means that it behaves like a vector except under a mirror-like inversion.

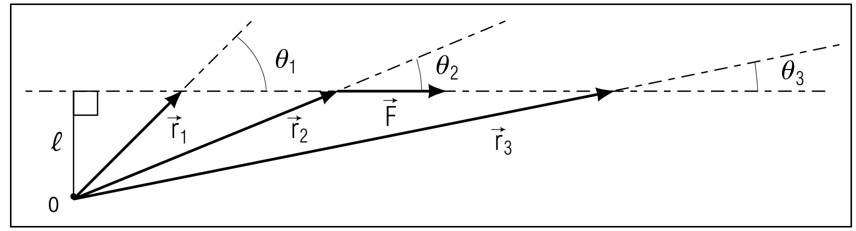

Figure 10. A force, $\vec{F}$, shown with three possible position vectors.

Using Fig. 10, we see that:

$$r_1\,\sin\theta_1 = r_2\,\sin\theta_2 = r_3\,\sin\theta_3 = \ell\,, \tag{10}$$

and hence from Eq. (9), any one of the three could be used. Simpler still is the equivalent expression:

$$\tau = \ell\,F\,. \tag{11}$$

Here ℓ, the perpendicular distance from point O to the line of action of the force, is called the force's "lever arm" or "moment arm." As a result, the magnitude of the torque is always equal to the product of the lever arm and the magnitude of the force.

3d. Torque from Two or More Forces. Since torque is a vector, if there are two or more torques relative to point O, the resultant torque about point O is just the vector sum of the individual torques. From Fig. 11,

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1$$

$$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2$$

yielding a resultant torque: $\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2$.

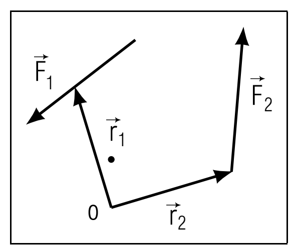

Figure 11. Two forces producing a resultant torque.

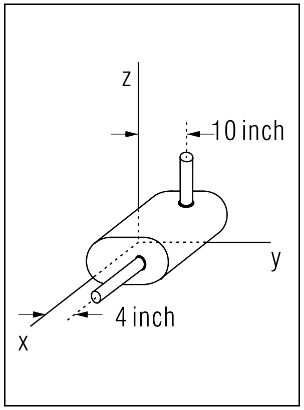

Figure 12. A gearbox.

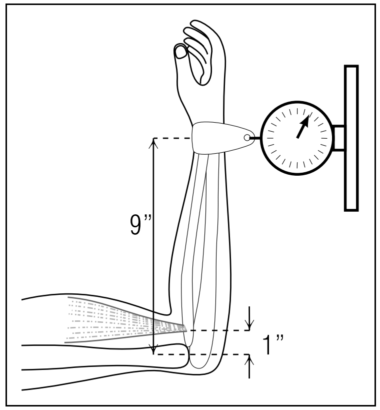

Figure 13. A test of strength.

3e. Example: A Gearbox. ▷ A reduction gearbox is shown in Fig. 12. The torques on it, about the point of intersection of the shafts, a point in the xz-plane, are $20\,\hat{z}$ ft lb and $4\,\hat{x}$ ft lb. Find the total resultant torque on the gearbox. (Ans. The vector sum.)

3f. Example: A Test of Strength. ▷ At a local count fair, a barker challenges you to the "game" shown in Fig. 13. You place your wrist in the strap, flex your arm, and try to exert a force of 200 lb on the spring scale. If the distance from your wrist to your elbow is 9 inches, find the torque exerted by the strap about your elbow and the force on your muscle. (Ans. 1800 in lb and 1800 lb.)

Acknowledgments

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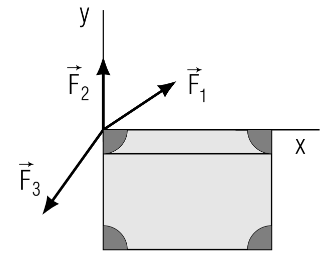
PROBLEM SUPPLEMENT

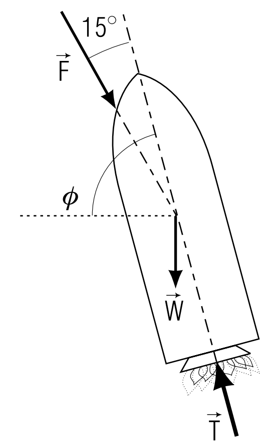
Problems 5, 9 and 10 also occur in this module's *Model Exam.*

1. You are pulling someone on a sled across a horizontal field. If you are exerting a force of 50 pounds directed 30° above the horizontal, what are the horizontal and vertical components of this force?

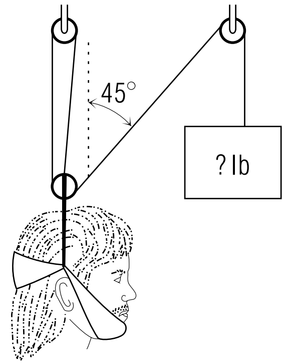
2. Three people pull on ropes attached to a single point on a trunk as shown in the figure. The ropes are in the same vertical xy-plane. The forces are: $F_1 =$ 10 lb at an angle of 30° with the x-axis; $F_2 = 25$ lb, upward; and $F_3 = 30$ lb at an angle of 220° with the x-axis. Find the x-component of the resultant force; the y-component of the resultant force; the magnitude and direction of the resultant force.



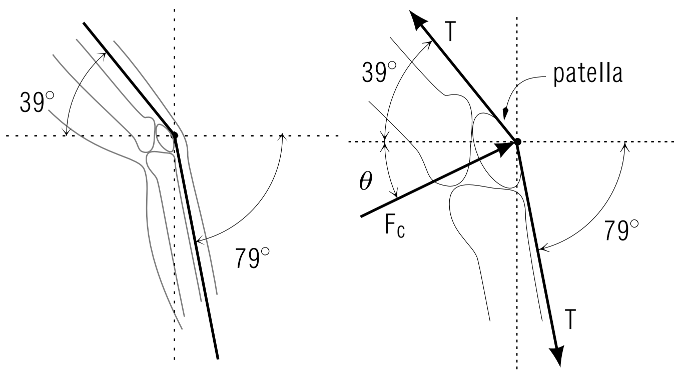
3. A Titan rocket booster (with payload) weighs 250,000 lb. The engines produce a total thrust of 400,000 lb. Due to air resistance there is a force of 10,000 lb acting at a 15° angle (as shown). If the angle of the thrust, from the horizontal (the dotted line in the sketch), is $\phi = 85°$, find the resultant force on the rocket. (Data from NASA).



4. A patient needs cervical traction to set a jaw fracture. The desired upward force is 6 lb. What weight is needed to produce this? Treat this problem as if all forces act at one point. Ignore the net horizontal force of the device.

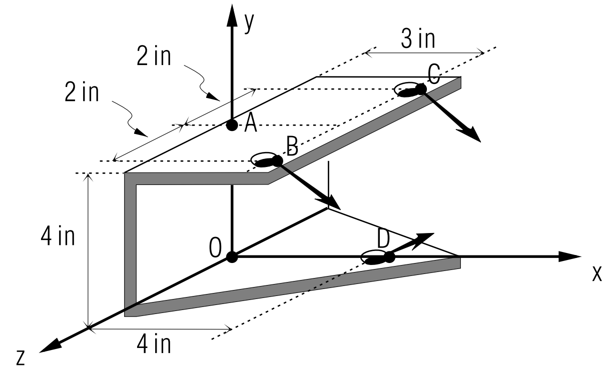


5. As the Quadriceps tendon is stretched over the patella (knee cap), it makes angles of 39 and 79 degrees with the horizontal. The tension in the tendon is 250 lb. Find the force exerted by the bones on the patella (F_c). (Data from the Michigan State University Dept. of Biomechanics).

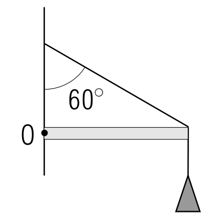


6. A coupling for a "semi" truck-trailer is shown in the sketch. The forces at B and C are directed along $(\cos 45°\hat{x} + \sin 45°\hat{z})$ and are 500 lb in magnitude. The force at D is directed along $(\cos 30°\hat{x} + \sin 30°\hat{y})$ and is 750 lb. in magnitude. Find the resultant force on the coupling, due to these three forces, and the resultant torque about the origin

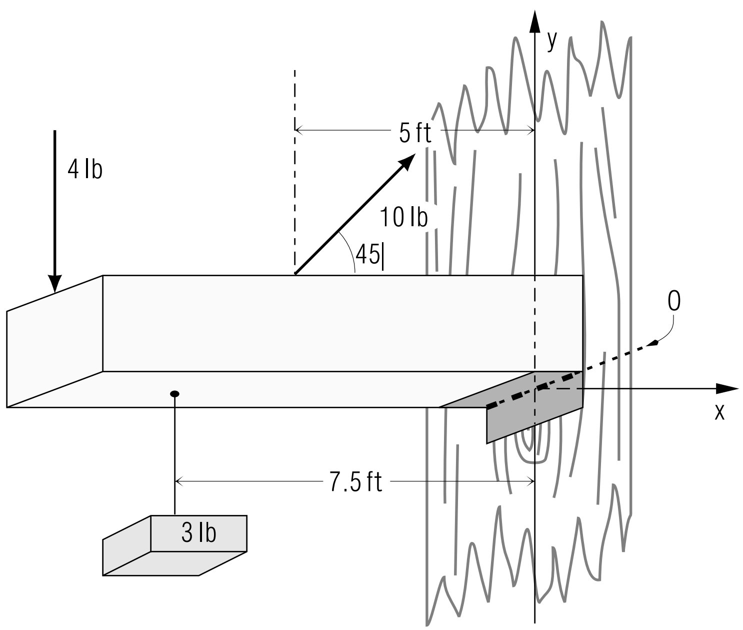
on it.



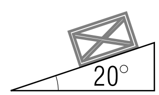
7. The beam and weight in the figure are supported by a cord that makes an angle of 60° with the wall. What torque does the cord produce on the beam about the point O if the tension in the cord is 100 N and the beam is 1.5 meters long?



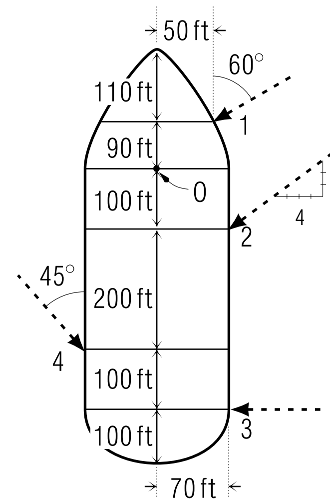
8. In the sketch, a uniform horizontal bar 10 ft long, weighing 5 lb, is hinged about a horizontal axis at O and is acted on by three additional forces. All forces are in the vertical plane. Find the total torque tending to rotate the bar about O. Assume the weight vector of the beam to act at its center.



9. A 1.0×10^2 lb crate rests on an inclined plane which makes an angle of 20.0° with the horizontal. What are the components of the weight parallel and perpendicular to the incline?

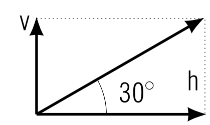


10. The Queen Elizabeth is being maneuvered into her berth in Los Angeles. Each of the four ocean-going tugs assisting her exerts a force of 5.0×10^4 lb. Find the resultant torque about the point O. (Data from C. Lyman, Naval Architect, South Bristol, Maine).



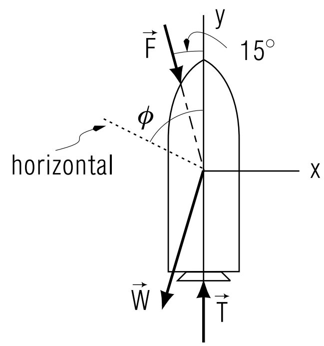
Brief Answers:

1. horizontal component $= (50\,\text{lb})(\cos 30°) = 43.3\,\text{lb}$
   vertical component $= (50\,\text{lb})(\sin 30°) = 25\,\text{lb}$.

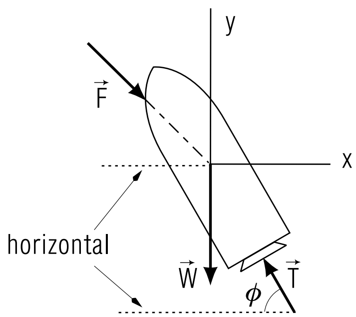


2. $\vec{F}_1 = 10\,\text{lb}(\cos 30°\hat{x} + \sin 30°\hat{y})$
   $\vec{F}_2 = 25\,\text{lb}(\hat{y})$
   $\vec{F}_3 = 30\,\text{lb}(\cos 220°\hat{x} + \sin 220°\hat{y})$
   $\sum \vec{F} = (-14.3\hat{x} + 10.7\hat{y})\,\text{lb}$, $|\vec{F}| = 17.9\,\text{lb}$; $\theta = 143.2°$

3. $\vec{T} = 400,000\,\text{lb}\hat{y}$
   $\vec{W} = 250,000\,\text{lb}\,[\cos(180° + \phi)\hat{x} + \sin(265°)\hat{y}]$
   $\vec{F} = 10,000\,\text{lb}\,[\cos(270° + 15°)\hat{x} + \sin(285°)\hat{y}]$
   $\vec{T} + \vec{W} + \vec{F} = (-1.92 \times 10^4\,\text{lb})\hat{x} + (1.41 \times 10^5\,\text{lb})\hat{y}$
   $|\vec{T} + \vec{W} + \vec{F}| = 1.42 \times 10^5\,\text{lb}$

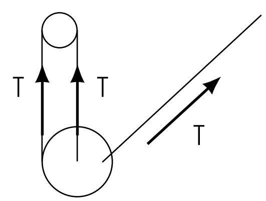


OR:



$\vec{T} = (400,000\,\text{lb})\,[\cos(180° - \phi)\,\hat{x} + \sin(95°)\,\hat{y}]$
$\vec{F} = (10,000\,\text{lb})\,[\cos(275° + 15°)\,\hat{x} + \sin(290°)\,\hat{y}]$
$\vec{W} = (250,000\,\text{lb})(-\hat{y})$
$\vec{T} + \vec{F} + \vec{W} = (-3.14 \times 10^4 \hat{x} + 39 \times 10^5 \hat{y})\,\text{lb}$
$|\vec{T} + \vec{F} + \vec{W}| = 1.42 \times 10^5\,\text{lb}$

4. $6\,\text{lb} = T(2 + \cos 45°)$
   $T = 2.22\,\text{lb} = W$



5. $\vec{T}_1 = (250\,\text{lb})\,[\cos(141°)\hat{x} + \sin(141°)\hat{y}]$

   $\vec{T}_2 = (250\,\text{lb})\,[\cos(-79°)\hat{x} + \sin(-79°)\hat{y}]$

   $\vec{T}_1 + \vec{T}_2 + \vec{F}_c = 0$
   $\vec{F}_c = -(\vec{T}_1 + \vec{T}_2) = (146.6\hat{x} + 88.1\hat{y})\,\text{lb}$

6. $\vec{F}_B = (500\,\text{lb})\,[\cos(45°)\hat{x} + \sin(45°)\hat{z}]$
   $\vec{F}_C = \vec{F}_B$
   $\vec{F}_D = (750\,\text{lb})\,[\cos(30°)\hat{x} + \sin(30°)]\hat{y}$
   $\sum \vec{F} = \vec{F}_B + \vec{F}_C + \vec{F}_D = 1357\,\text{lb}\hat{x} + 375\,\text{lb}\hat{y} + 707\,\text{lb}\hat{z}$
   $\vec{r}_B = 3\,\text{in}\hat{x} + 4\,\text{in}\hat{y} + 2\,\text{in}\hat{z}$
   $\vec{r}_C = 3\,\text{in}\hat{x} + 4\,\text{in}\hat{y} - 2\,\text{in}\hat{z}$
   $\vec{r}_D = 4\,\text{in}\hat{x}$

$$\vec{\tau}_B = \vec{r}_B \times \vec{F}_B = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & 4 & 2 \\ 353.6 & 0 & 353.6 \end{vmatrix} \text{in lb}$$
$$= [(1414)\hat{x} + (-353.6)\hat{y} + (-1414)\hat{z}]\ \text{in lb}$$
$$\vec{\tau}_C = \vec{r}_C \times \vec{F}_C = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 3 & 4 & -2 \\ 353.6 & 0 & 353.6 \end{vmatrix} \text{in lb}$$
$$= [(1414)\hat{x} + (-1768)\hat{y} + (-1414)\hat{z}]\ \text{in lb}$$

   $\vec{\tau}_D = 4\,\text{in}\hat{x} \times (750\,\text{lb})(\sin 30°\hat{y}) = (1500\,\text{in lb})\hat{z}$
   $\sum \vec{\tau} = (2828\,\text{in lb})\hat{x} + (-2121\,\text{in lb})\hat{y} + (-1328\,\text{in lb})\hat{z}$

7. $\vec{\tau} = \vec{r} \times \vec{F}$
   $= (1.5\,\text{m})\hat{x} \times (100\,\text{N})[\cos(150°)\hat{x} + \sin(150°)\hat{y}]$
   $= (75\,\text{N m})\hat{z}$ (out of page)

8. $\sum \vec{\tau} = r_A F_A \hat{z} + r_{W_3} W_3 \hat{z} + r_T T \sin(45°)\hat{z} + \left(\dfrac{\ell}{2}\right)(W_B)\hat{z}$
   $= (-10\,\text{ft})(-4\,\text{lb})\hat{z} + (-7.5\,\text{ft})(-3\,\text{lb})\,\hat{z}$
   $+ (-5\,\text{ft})(10\,\text{lb})(\sin 45°)\hat{z} + (-5\,\text{ft})(-5\,\text{lb})\hat{z}$
   $= 52\,\text{ft lb}\hat{z}$

9. 93.97 lb perpendicular and 34.20 lb parallel

10. $-1.01 \times 10^7\,\text{ft lb}\hat{z}$

SPECIAL ASSISTANCE SUPPLEMENT

1. A Look Back at Understanding Force

1a. Statement and example. A force is a push or pull which alone would cause an object to accelerate. For instance, if when you push on a closed but unfastened, door expect it to acquire a velocity. Your push remains a force even if the door is latched and so does not move.

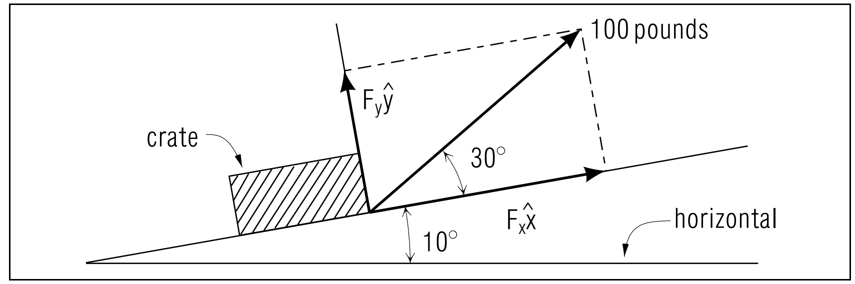


Figure 14. Resolution of the force acting on the crate into components.

1b. Properties. Force is a vector quantity and hence is not specified unless we know both its magnitude and direction. An arrow over a letter, such as $\vec{F}$, is the symbol for a vector. In the same context, the letter standing alone, F, or the vector symbol confined in bars $|\vec{F}|$, represents the magnitude of the vector $\vec{F}$. An arrow is used to represent a force graphically. Forces may be added vectorially and/or resolved into components. The resultant of several given forces has components which are exactly equal to the sum of the components of the original forces. Common units of force are newtons (N) and pounds (lb).

1c. Example 1. Determine which of these statements completely specifies a force:

a. $\vec{F}$ (on a diagram)

b. F (on a diagram)

c. 10 pounds acting vertically upward

d. 10 pounds acting horizontally

e. magnitude and direction

Answers: a, c and e

1d. Example 2. A crate is being pulled up a ramp which makes an angle of 10.0° with the horizontal. If the force acting is 1.00×10^2 pounds directed 30° above the ramp, what are the components of this force parallel to the ramp and perpendicular to the ramp?

Solution: First construct a drawing (see Fig. 14), to be sure that we understand the problem. To use regular x,y components, superimpose an

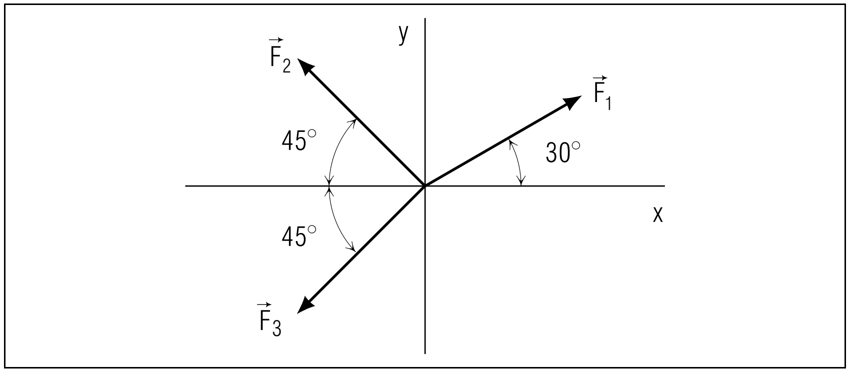


Figure 15. Force vectors to be added.

x,y coordinate system such that the origin is at the point of application of the force, the x-axis is directed up the ramp, and the y-axis is perpendicular to the ramp. The component parallel to the ramp is $\mathrm{F}_x\hat{x}$ and the component perpendicular to the ramp is $\mathrm{F}_y\hat{y}$.

Using the right triangles formed, we can write:

$$F_x = (100.\,\mathrm{lb})\cos 30^\circ = 86.6\,\mathrm{lb}, \qquad F_y = (100\,\mathrm{lb})\sin 30^\circ = 50.0\,\mathrm{lb},$$

where F_x and F_y are the magnitudes of the components.

The vector components are:

$F_x\hat{x} = 86.8\hat{x}$ pounds (directed up the ramp in the positive x-direction);

$F_y\hat{y} = 50.0\hat{y}$ pounds (directed perpendicular to the ramp in the positive y-direction).

Three significant figures are kept because the values given in the problem had three significant figures.

1e. Example 3. Add the three forces shown in Fig. 15 by using vector components. Express your final answer by giving the magnitude and direction of the resultant. Let: $F_1 = 1.00 \times 10^2\,\mathrm{N}$, $F_2 = 8.0 \times 10^1\,\mathrm{N}$, $F_3 = 8.0 \times 10^1\,\mathrm{N}$.

First, recall the properties of vector addition using vector components. The resultant will be determined by equations

$$\begin{aligned} \sum F &= \left(\sum F_x^2 + \sum F_y^2\right)^{1/2}, \\ \tan\theta &= \frac{\sum F_y}{\sum F_x}, \end{aligned} \tag{1}$$

where

$$\begin{aligned} \sum F_x &= F_{1x} + F_{2x} + F_{3x}, \\ \sum F_y &= F_{1y} + F_{2y} + F_{3y}. \end{aligned} \tag{2}$$

Therefore, we begin by finding the components of the given forces and then add these to find the components of the resultant.

$$\begin{aligned} F_{1x} &= F\cos 30^\circ = (100\,\mathrm{N})(0.866) = 86.6\,\mathrm{N} \\ F_{1y} &= F_1\sin 30^\circ = (100\,\mathrm{N})(0.500) = 50.0\,\mathrm{N} \\ F_{2x} &= -F_2\cos 45^\circ = -(80\,\mathrm{N})(0.707) = -57\,\mathrm{N} \end{aligned} \tag{3}$$

$$\begin{aligned} F_{2y} &= F_2\sin 45^\circ = (80\,\mathrm{N})(0.707) = 57\,\mathrm{N} \\ F_{3x} &= -F_3\cos 45^\circ = -(80\,\mathrm{N})(0.707) = -57\,\mathrm{N} \\ F_{3y} &= -F_3\sin 45^\circ = -(80\,\mathrm{N})(0.707) = -57\,\mathrm{N} \end{aligned}$$

The negative sign indicate that these particular components are in either the negative x or negative y directions.

Hence, for example, $F_{2x}\hat{x} = -57\,\mathrm{N}\hat{x} = 57\,\mathrm{N}\,(-\hat{x})$ means a 57 newton force in the negative x-direction. Now, if we add the x-components algebraically, we get

$$\sum F_x = (86.6 - 57 - 57)\,\mathrm{N} = -27.4\,\mathrm{N}.$$

Similarly

$$\sum F_y = 50\,\mathrm{N}.$$

Note that the Eqs. (2) written in symbolic form have all positive signs.

It is only when we substitute in numbers that we introduce the negative signs to indicate the negative direction [Eqs. (3)].

Figure 16 shows these components and their resultant, $\sum \vec{F}$.

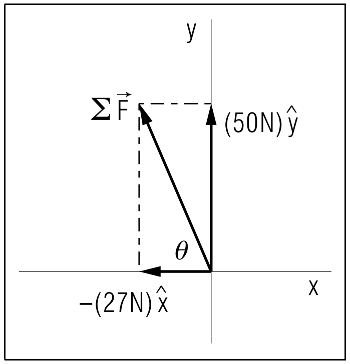


Figure 16. The components and resultant of the forces shown in Fig. 15.

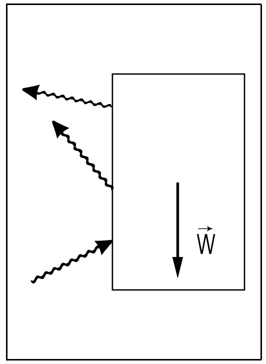


Figure 17. Force diagram for a three hinged door.

Using Eqs. (1),

$$\left|\sum \vec{f}\right| = \left[(-27.4)^2 + (50)^2\right]^{1/2}\ \text{N}$$
$$= 57\,\text{N}\,,$$

and

$$\tan\theta = \frac{50\,\text{N}}{27.4\,\text{N}}\,;$$
$$\theta = \tan^{-1}\left(\frac{50\,\text{N}}{27.4\,\text{N}}\right) = 61\,.$$

Hence $\sum \vec{F}$ is 57 N directed 61° above the negative x-axis.

We have retained only two significant figures in the resultant because the data had only two significant figures in most cases.

2. Understanding a Force Diagram

2a. Statement and Example. A force diagram is a drawing showing an isolated object and all forces acting on it. For example, an ordinary door which is hung by three hinges has four forces acting on it. The force diagram is shown in Fig. 17. Note that the forces exerted by the hinges are represented by wavy arrows with round tips to indicate unknown magnitudes and directions.

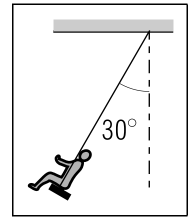


Figure 18. Child on a Swing.

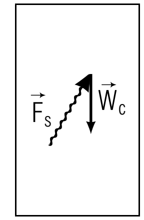


Figure 19. Force diagram for the child.

2b. Applicability. In studying a force system for the possible determination of some unknown forces, we begin by constructing a force diagram. Actual calculations of unknown forces are carried out in Sect. 3, the next section.

2c. Problem Example. A child of known weight is swinging on a swing which consists of a swing board supported by two ropes. A side view is shown in Fig. 18 where the two ropes are treated as one for consideration of motion in the plane of the swinging.

When the swing is located at 30°, draw two force diagrams, one for the child and one for the seat of the swing.

Assume that the ropes are straight and that they do not exert any force directly on the child. Under the conditions given, the child experiences only two forces: weight, $\vec{W}_c$, and the force exerted by the seat, $\vec{F}_s$. The weight has a known magnitude and direction while the seat force has unknown direction and magnitude. Figure 19 shows the force diagram for the child.

For the swing seat there are three forces acting: the force exerted by the child, $\vec{F}_c$, the force exerted by the ropes, F_r, and the weight of the seat, $\vec{W}_s$. The forces, $\vec{F}_r$ and $\vec{W}_s$ have known directions. $\vec{F}_c$ has unknown magnitude and direction. Figure 20 shows the force diagram for the swing seat.

The forces $\vec{F}_s$ and $\vec{F}_c$ discussed above are action-reaction forces between the child and the seat. Recall that a single isolated force cannot exist: every force must have a partner that has the same magnitude but is in

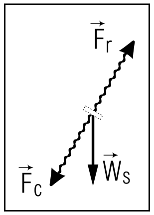


Figure 20. Force diagram for the seat.

the opposite direction. This application of Newton's third law of motion tells us that: $\vec{F}_s = -\vec{F}_c$.

/AsSect

3. Understanding Equilibrium

3a. Statement and Example. The only condition necessary for the static equilibrium of a point acted upon by forces or for the static equilibrium of an extended, rigid object acted upon by concurrent forces is that the vector sum of the forces on the object is zero. (There will be no need to use rotational considerations in the cases included in this section.)

For example, a picture with a cord attached near each side is hung by catching the cord over a nail on the wall (see Fig. 21). The nail is in equilibrium. The picture is an extended object which is in equilibrium under the action of concurrent forces. These forces are not acting at a point but have lines of action that intersect at a common point, the nail.

3b. Interpretation. Figure 21 shows a view from behind the picture. There are three concurrent forces acting on the picture: two of the forces are exerted by the cord and the third by the weight. (Any reaction of the wall directly on the picture itself has been neglected.)

The force diagram showing the concurrent forces is presented in Fig. 22. The vector equilibrium equation is:

$$\vec{F}_A + \vec{F}_B + \vec{W} = 0\,. \tag{4}$$

3c. Problem Solution. The algebraic equations which we would use to solve for the unknown magnitudes of $\vec{F}_a$ and $\vec{F}_b$, given the weight of the picture, $W = 10$ pounds, are components of Eq. (4),

$$\begin{aligned} x\text{-dir: } & F_a \cos 50^\circ - F_b \cos 50^\circ = 0, \\ y\text{-dir: } & F_a \sin 50^\circ + F_b \sin 50^\circ - 10\,\text{lb} = 0\,. \end{aligned} \tag{5}$$

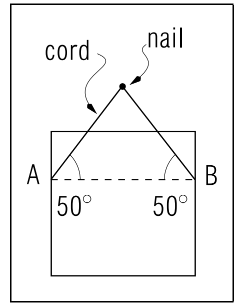


Figure 21. A hanging picture.

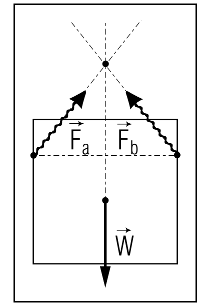


Figure 22. Force diagram for picture.

(Remember that when putting the selected information from the force diagram into the component equations we use negative signs to indicate negative $x-$ or y-directions.) From the first of Eqs. (5), $F_a = F_b$ and using this in the second equation yields

$$F_a \sin 50^\circ + F_a \sin 50^\circ - 10\,\text{lb} = 0\,,$$

or

$$F_a = \frac{10\,\text{lb}}{2 \sin 50^\circ} = 6.5\,\text{lb}\,.$$

Since $F_a = F_b$, we get: $F_b = 6.5$ pounds.

4. Understanding Torque

4a. Statement and Example. Torque is defined as the cross product of two vectors, $\vec{r} \times \vec{F}$. As an example consider exerting a 500 newton force on a flag pole as shown in Fig. 23. The torque about the base of the pole produced by this force has a magnitude

$$\tau = rF \sin\theta\,.$$

A diagram showing the force $\vec{F}$ and the position vector $\vec{r}$ is given in Fig. 24. The magnitude of $\vec{F}$ is 500 newtons and the magnitude of $\vec{r}$ is3.0 meters. Then the magnitude of the torque is

$$\begin{aligned} \tau &= (3.0\,\text{m})(500\,\text{N}) \sin 120^\circ \\ &= (3.0\,\text{m})(500\,\text{N})(0.866) \\ &= 1.3 \times 10^3\,\text{N m}\,. \end{aligned}$$

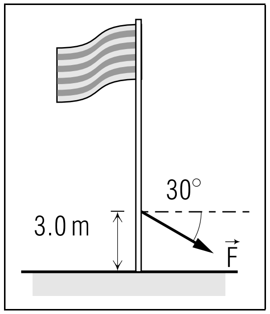


Figure 23. Flagpole with force acting.

4b. Properties. Torque is a vector quantity with units of newton-meter or pound-feet. The common symbol is the Greek letter τ. The direction of $\vec{\tau}$ is determined by the right-hand rule. For example, in Figs. 23 and 24 the torque vector is directed into the page but this actually means a tendency to rotate clockwise. (A torque directed into the page does not mean that the flagpole tends to turn into the page.)

4c. Interpretation. The position vector, $\vec{r}$, runs from the reference point O to any point on the line of action of the force (not necessarily to the point of application of the force). The magnitude of $\vec{\tau}$ is $rF\sin\theta$, where r is the magnitude of the position vector, F is the magnitude of the force, and θ is the angle between the vectors $\vec{F}$ and $\vec{r}$. When the directions of $\vec{r}$ and $\vec{F}$ are fixed ($\theta =$ constant) the magnitude of the torque is directly proportional to r and F. (These are magnitudes.)

For example if $\vec{F}$ is constant but is moved twice as far from the reference point O as it was originally, the torque will double. Or if $\vec{r}$ is constant and the force is doubled, the torque will double. The magnitude of the torque can also be written as $\tau = hF$, where $h(= r\sin\theta)$ is the lever arm, or as $\tau = rF_\perp$, where $F_\perp$ is the component of $\vec{F}$ perpendicular to $\vec{r}$. The lever arm is the perpendicular distance from the reference point O to the line of action of the force. Hence another view of torque is that for a given force the torque is directly proportional to the lever arm.

4d. Comparison. Force and torque are both vector quantities. For a system of forces acting on the body, the resultant force tells if the body has a translational acceleration while the resultant torque tells if the body has a rotational acceleration.[4]

[4] See "Non-Concurrent Forces; Centers of Force, Gravity, Mass" MISN-0-6.

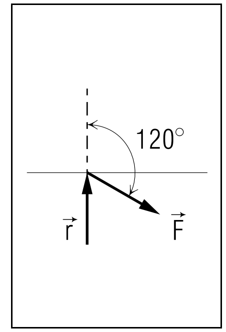


Figure 24. Diagram of force and position vector.

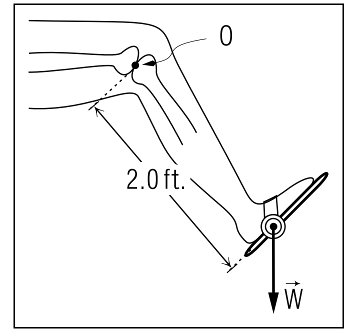


Figure 25. Lower leg, foot, and deLorme boot with weight.

4e. Problem Example. You have just torn the cartilage in your knee as the result of a skiing accident. What is in store for you? You may have surgery to remove the cartilage and then as soon as possible begin exercises to restore your leg to full strength and normal functioning. The exercise can be done by using a DeLorme boot and weights. In Fig. 25 assume the total weight, W of the boot assembly to be 40 pounds. Calculate the torque produced by the weight about your knee (reference point O) when

a. your foot hangs straight down,

b. your foot is raised to the point where your lower leg makes a 45° angle with the horizontal and

c. your lower leg is extended horizontally.

We are only concerned with the weight W and reference point O located 2.0 feet away. Figure 26 shows the three cases in question.

To calculate torque, we begin by looking at the definition of torque,

$$\vec{\tau} = \vec{r} \times \vec{F}\,.$$

Part a. Foot hangs straight down (Fig. 26a):

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{W}\,.$$

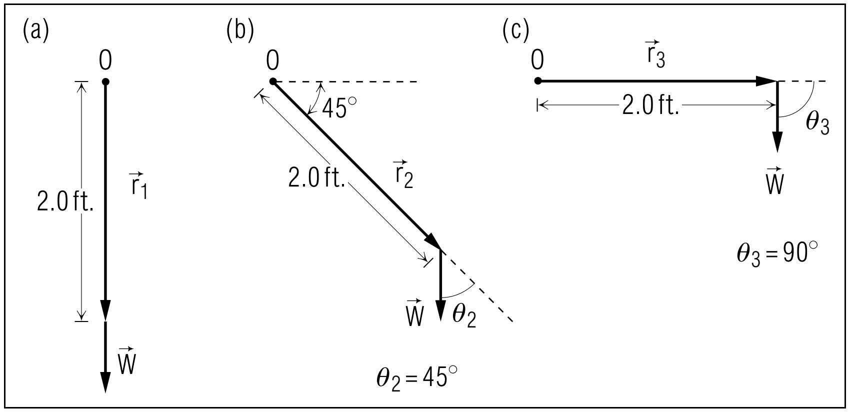


Figure 26. Force diagram, three leg positions.

The magnitude of $\vec{\tau}_1$ is $\tau_1 = r_1 W \sin\theta$. But in this case, $\vec{r}_1$ and $\vec{W}$ have the same direction and hence the angle between then is zero degrees. The result is $\tau_1 = 0$, because the $\sin 0° = 0$.

Part b. Raised 45° (Fig. 26b):

$$\vec{\tau}_2 = \vec{r}_2 \times \vec{W}\,,$$

with magnitude

$$\tau_2 = r_2 W \sin\theta_2 = (2.0\,\text{ft})(40\,\text{lb})\sin 45° = (2.0)(40)(0.707)\,\text{ft lb} = 57\,\text{ft lb}\,.$$

The direction of $\vec{\tau}_2$ is into the page (use the right-hand rule).

Part c. Horizontal (Fig. 26c):

$$\vec{\tau}_3 = \vec{r}_3 \times \vec{W}\,,$$

with magnitude

$$\tau_3 = r_3 W \sin\theta_3 = (2.0\,\text{ft})(40\,\text{lb})\sin 90° = 80\,\text{ft lb}\,.$$

The direction of τ_3 is also into the page.

The implications of this for determining muscle strength can be considered following a discussion of rotational equilibrium.[5]

[5] See “Static Equilibrium, Centers of Force, Gravity and Mass” (MISN-0-6).

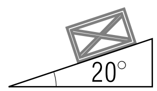
S-1 *(from TX-2d)*

Analyze the forces exerted on the pulley attached to the bottom of the foot: they should add to zero so the foot doesn’t accelerate off somewhere (that would be quite a surprise to the patient!).
The tensions in the rope are just W, the Weight’s weight.
The force exerted by the leg on the pulley is 22 pounds, in some unknown direction below the horizontal (we can designate it by some symbol). Thus there are forces acting in three directions on the pulley. They should add to zero for static equilibrium. Note that static equilibrium will occur unless the leg is pulled off the patient: what we are solving for is static equilibrium with the angles and leg-force desired.

MODEL EXAM

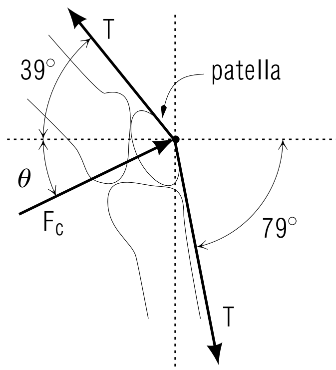
1. See Output Skills K1-K2 in this module's *ID Sheet*.

2.

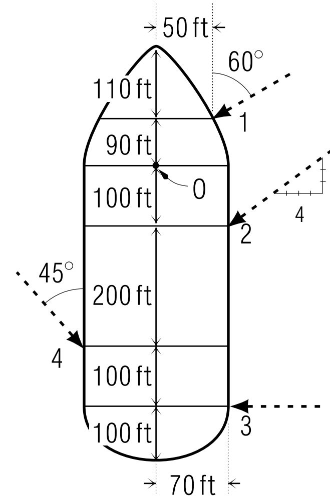


A 1.0×10^2 pound crate rests on an inclined plane that makes an angle of 20.0° with the horizontal. What are the components of the weight parallel and perpendicular to the incline?

3. As the Quadriceps tendon is stretched over the patella (knee cap), it makes angles of 39 and 79 degrees with the horizontal. The tension in the tendon is 250 lb. Find the force exerted by the bones on the patella (F_c). (Data from the Michigan State University Dept. of Biomechanics).



4.



The Queen Elizabeth is being maneuvered into her berth in Los Angeles. Each of the four ocean-going tugs assisting her exerts a force of 5.0×10^4 lb. Find the resultant torque about the point O. (Data from C. Lyman, Naval Architect, South Bristol, Maine).

Brief Answers:

1. See this module's *text*.
2. See this module's *Problem Supplement*, problem 9.
3. See this module's *Problem Supplement*, problem 5.
4. See this module's *Problem Supplement*, problem 10.